Discretisations of Rough Stochastic Partial Differential Equations

by

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Contents

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Declarations

This thesis is submitted to the University of Warwick in support of my application for the degree of Doctor of Philosophy. I declare that this thesis has not been submitted in any previous application for any degree and it contains the material which is my own original work, unless otherwise stated, cited or commonly known. Parts of this thesis have been published or submitted for publication in [HM15a] and [HM15b].

Abstract

This thesis consists of two parts, in both of which we consider approximations of rough stochastic PDEs and investigate convergence properties of the approximate solutions. In the first part we use the theory of (controlled) rough paths to define a solution for one-dimensional stochastic PDEs of Burgers type driven by an additive space-time white noise. We prove that natural numerical approximations of these equations converge to the solution of a corrected continuous equation and that their optimal convergence rate in the uniform topology (in probability) is arbitrarily close to $\frac{1}{2}$. In the second part of the thesis we develop a general framework for spatial discretisations of parabolic stochastic PDEs whose solutions are provided in the framework of the theory of regularity structures and which are functions in time. As an application, we show that the dynamical Φ^4_3 model on the dyadic grid converges after renormalisation to its continuous counterpart. This result in particular implies that, as expected, the Φ^4_3 measure is invariant for this equation and that the lifetime of its solutions is almost surely infinite for almost every initial condition.

Chapter 1

Introduction

Stochastic PDEs are used to describe many physical, biological and economical systems which, in contrast to deterministic systems, are subject to a random "noise", see e.g. [BS95, GLP99, HL09]. This randomness can come from both intrinsic sources, like some inherent features of the models, and extrinsic sources, like environmental influences. In many cases, presence of a random noise is described by adding a quite irregular extra term to an equation, which affects the solutions by decreasing their regularities. Very often low regularities of terms in an equation cause a problem already on the level of *defining a solution*, not to mention investigating any useful properties of the model.

A typical parabolic stochastic PDE on the time-space domain $\mathbb{R}_+ \times \mathbb{R}^d$ with $d \geq 1$ has the form

$$\partial_t u = Au + F(u, \xi) , \tag{1.1}$$

where A is an elliptic differential operator, F is a non-linear term and ξ is a random noise. Such equation should be equipped with some initial condition at $t = 0$, and the term F can in principle depend not only on u itself, but also on its derivatives of the orders strictly smaller than those in A. A large class of such equations has been solved and analysed via e.g. the stochastic integration theory [DPZ14] or the theory of Dirichlet forms [AR91]. In special cases there have been also attempts to consider stochastic PDEs beyond the scope of these techniques, e.g. using Wick products in place of the standard ones in [DPD03], using the Cole-Hopf transform in [BG97] or using white noise analysis in [HØUZ10]. However, these attempts have been lacking either a systematic approach to a larger class of equations, or produced non-physical solutions whose analysis did not seem to be reasonable [Cha00]. A systematic and theoretically reasonable treatment of the equations when the term $F(u, \xi)$ cannot be a priori defined in a classical way (e.g. when the noise ξ is an irregular distribution so that u is expected to be a distribution as well, and $F(u, \xi)$ contains a power of u) and the standard approaches cannot be applied have been an open question for a long time. In what follows, we refer to such equations which cannot be treated classically as "*rough stochastic PDEs*".

Thanks to several recent breakthroughs in the fields of stochastic ODEs and PDEs by T. Lyons and M. Hairer, it has become possible to give a systematic notion of (at least local in time) solutions for a large class of rough stochastic equations by the theories of *rough paths* [Lyo98] and *regularity structures* [Hai14], including stochastic Burgers-type equations, the KPZ equation, the Φ^4_3 equation, the $2D$

parabolic Anderson model and the Navier-Stokes equations in 2 and 3 dimensions [ZZ15b]. In several cases further analysis of these solutions, e.g. numerical approximations, global well-posedness in time or investigation of a certain limiting behaviour, proved to be possible [HMW14, HM15a, HQ15]. We provide below a brief overview of these two theories and their usage in this work for analysing the limits of approximate equations to a large class of rough stochastic PDEs.

At the same time, the theory of rough paths served as a motivation for development of the *paracontrolled calculus* [GIP15] which gives an alternative approach to some of the rough equations covered by the regularity structures, for example the Φ^4_3 equation [CC13, ZZ15a], the $3D$ Navier-Stokes equation [ZZ14] and the KPZ equation [GP15]. Although, in contrast to the regularity structures, the paracontrolled calculus lacks some generality at this stage, we expect that one can show a certain correspondence between the two theories.

1.1 Analysis of stochastic PDEs using rough paths

In the celebrated work [Lyo98], T. Lyons introduced a new approach, called *rough paths theory*, to the controlled equations of the form

$$dY_t = F(Y_t)\, dX_t\ , \tag{1.2}$$

where $X, Y : [0, T] \to \mathbb{R}^n$ are two paths of Hölder regularities $\alpha \in \big(0, \frac{1}{2}\big]$ and $F : \mathbb{R}^n \to \mathbb{R}$ is a smooth function. The aim of the author was to find a solution to this equation without using stochastic integration, but working rather in a pathwise manner. The idea of rough paths theory is to lift the path X to another path $\mathbf{X}$ living in a much higher dimensional space and containing information about the iterated integrals of X with respect to itself. The integral in (1.2) is then given as a limit of modified Riemann sums, defined by the whole path $\mathbf{X}$, see (2.17) below.

M. Gubinelli extended this idea in [Gub04, Gub10] by noticing that in order to define the integral

$$\int_s^t Y_r \otimes dX_r\ ,$$

one only needs to know that the path Y "behaves similarly" to X, in the sense that small increments of Y are close in some sense to the respective increments of X.

More precisely, there exist functions $Y' : [0,T] \to \mathbb{R}^{n\times n}$ and $R : [0,T]^2 \to \mathbb{R}^n$ of respective regularities α and 2α (here, the notation Y' does not mean a derivative in the classical sense) such that for any $s, t \in [0,T]$ one has

$$Y_t - Y_s = Y'_s\big(X_t - X_s\big) + R(s,t)\ .$$

In this case, one says that such process Y is *controlled* by X, and one calls this extension *the theory of controlled rough paths*.

The idea of M. Gubinelli was used by M. Hairer in [Hai11] to give a notion of a (local) solution to the rough stochastic Burgers-type equation in one spatial dimension of the form

$$\partial_t u = \Delta u + F(u) + G(u)\,\partial_x u + \xi\ , \qquad u(0) = u^0\ , \tag{1.3}$$

where the solution u is periodic in space and is $\mathbb{R}^n$-valued, Δ is the Laplace operator, the two functions $F : \mathbb{R}^n \to \mathbb{R}^n$ and $G : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ are sufficiently smooth, ξ is a spatially periodic space-time white noise [DPZ02] and u^0 is an initial value. Due to the fact that the solution to the linearised equation

$$\partial_t X = \Delta X + \xi\ , \tag{1.4}$$

has almost surely Hölder continuity *strictly* lower than $\frac{1}{2}$, see [DPZ02, Hai09], the product even in $G(X)\,\partial_x X$ cannot be in general defined in a classical way, as a product of a function and a distribution [BCD11]. The key idea of [Hai11] was to test the nonlinearity with a smooth test function φ and to formally rewrite it as

$$\int_{\mathbb{R}} \varphi(x) G(u(t,x))\,\partial_x u(t,x)\,dx = \int_{\mathbb{R}} \varphi(x) G(u(t,x))\,d_x u(t,x)\ . \tag{1.5}$$

Moreover, it was noticed in [Hai11] that a solution u is expected to behave locally in space as a solution of the linearised equation (1.4). This correctly suggests that the theory of controlled rough paths could be used to deal with the integral (1.5) in the pathwise sense. In general, if we don't require the functions F and G to be bounded together with their derivatives, the solution u can be defined only locally in time. A similar idea has been used to give a notion of solution of the Burgers-type equation with a multiplicative noise (when the last term in (1.3) is replaced by $\theta(u)\xi$, for a sufficiently regular nonlinear local function θ) in [HW13] and of the KPZ equation in [Hai13].

Using the notion of solution obtained by this approach, natural numerical approximations of equation (1.3) and its generalisations have been studied in [HM12, HMW14]. In general, the considered approximate equations were of the form

$$\partial_t u_\varepsilon = \Delta_\varepsilon u_\varepsilon + F(u_\varepsilon) + G(u_\varepsilon)\, D_\varepsilon u_\varepsilon + \xi_\varepsilon \;, \qquad u_\varepsilon(0) = u_\varepsilon^0 \;, \tag{1.6}$$

parametrised by $\varepsilon \in (0, 1]$, where the operators Δ_ε and D_ε approximate Δ and ∂_x respectively (e.g. they can be finite difference approximations), ξ_ε is an approximation of the noise ξ (for example, it can be given by a cut-off of the Fourier modes with high frequencies), and u_ε^0 is an initial value. One can find precise formulations of the approximate equations with examples in Section 2.2. In particular, it was shown that in the case when $u_\varepsilon^0 \to u^0$ sufficiently quickly as $\varepsilon \to 0$ in a Hölder space of a sufficiently low regularity and under quite general assumptions on the approximate equations (1.6), their solutions converge and that the convergence rate in the uniform topology (in probability) is arbitrarily close to $\frac{1}{6}$. More precisely, there exists a function $\bar{u}$, which is a local solution to a modified equation (1.3) on a random time interval $[0, T^*)$, and there exists a family of stopping times T_ε satisfying $\lim_{\varepsilon \to 0} T_\varepsilon = T^*$ in probability such that

$$\lim_{\varepsilon \to 0} \mathbb{P} \Big[\sup_{t \in [0, T_\varepsilon]} \|(\bar{u} - u_\varepsilon)(t)\|_{L^\infty} \geq \varepsilon^{\frac{1}{6} - \alpha} \Big] = 0 \;, \tag{1.7}$$

for every $\alpha > 0$. Another important point in these works was that different approximations of the Burgers-type equations (1.6) converge to different limits $\bar{u}$ which defer by an additional correction term appearing in (1.3), which was calculated explicitly. This extra term is a generalisation of the Itô-Stratonovich correction in the classical theory of SDEs.

1.2 Solving stochastic PDEs by regularity structures

The theory of *regularity structures* introduced in [Hai14] is a far-reaching generalisation of the theory of controlled rough paths. Its aim is to develop a systematic approach to formulating, solving and analysing solutions of rough stochastic PDEs of the form (1.1). The main assumption on the equations considered by the theory is *local subcriticality*, which roughly speaking means that if we rescale the equation in a way that keeps both the linear part and the noise ξ invariant, then at small scales the nonlinear terms formally disappear.

A "naïve" approach to the equation (1.1) is to consider a sequence of regularised equations given by

$$\partial_t u^\varepsilon = A u^\varepsilon + F(u^\varepsilon, \xi^\varepsilon) , \tag{1.8}$$

parametrised by $\varepsilon > 0$, where ξ^ε is a smoothened version of ξ (e.g. a mollification of ξ), and to show that u^ε converges to some limit u as $\varepsilon \to 0$ which is independent of the smoothening. This approach unfortunately in general fails when the noise ξ is irregular, even under the assumption of local subcriticality, giving infinite or trivial limits of the approximate solutions u^ε. This problem primarily appears due to the fact that the power u^k with $k > 0$ is not well defined if u is a distribution (generalised function).

A particular example prototypical for the class of equations we are interested in is the dynamical Φ^4 model in dimension 3, see [PW81], which can be formally described by the equation

$$\partial_t \Phi = \Delta \Phi - \Phi^3 + \xi , \qquad \Phi(0, \cdot) = \Phi_0(\cdot) , \tag{1.9}$$

on the torus $\mathbb{T}^3 \stackrel{\text{def}}{=} (\mathbb{R}/\mathbb{Z})^3$ in space, where Δ is the Laplace operator on $\mathbb{T}^3$, ξ is a space-time white noise over $L^2(\mathbb{R} \times \mathbb{T}^3)$ and Φ_0 is an initial value. If we formally perform the diffusive rescaling of this problem by writing

$$\tilde{\xi}^\varepsilon(t, x) = \varepsilon^{\frac{5}{2}} \xi(\varepsilon^{-2} t, \varepsilon^{-1} x) , \qquad \tilde{\Phi}^\varepsilon(t, x) = \varepsilon^{\frac{1}{2}} \Phi(\varepsilon^{-2} t, \varepsilon^{-1} x) ,$$

for some small parameter $\varepsilon > 0$, then we have on the one hand that $\tilde{\xi}^\varepsilon$ equals to ξ in distribution, and on the other hand $\tilde{\Phi}^\varepsilon$ solves

$$\partial_t \tilde{\Phi}^\varepsilon = \Delta \tilde{\Phi}^\varepsilon + \varepsilon (\tilde{\Phi}^\varepsilon)^3 + \tilde{\xi}^\varepsilon .$$

Formally the nonlinear term in the equation above vanishes as $\varepsilon \to 0$, implying local subcriticality of the problem. On the other hand, if we simply replace ξ in (1.9) by its mollified version ξ^ε and pass $\varepsilon \to 0$, the limit will vanish, see [HRW12] in the two dimensional case. However, in this case one can perform a certain *renormalisation* by adding a diverging constant $C^{(\varepsilon)}$ (which is called a *renormalisation constant*) to the right-hand side of the smooth equation so that the limit of the solutions to

$$\partial_t \Phi^\varepsilon = \Delta \Phi^\varepsilon + \big(C^{(\varepsilon)} \Phi^\varepsilon - (\Phi^\varepsilon)^3\big) + \xi^\varepsilon$$

is non-trivial, see [DPD03] for the $2D$ problem. This prompts that a "more correct" version of the continuous equation (1.9) is

$$\partial_t \Phi = \Delta \Phi + \infty\, \Phi - \Phi^3 + \xi\;, \qquad \Phi(0, \cdot) = \Phi_0(\cdot)\;, \tag{1.10}$$

where the "infinite constant" ∞ refers to the limit of the renormalisation constant $C^{(\varepsilon)}$. Usually, there is a certain freedom in the choice of renormalisation constants so that they form a *renormalisation group*. The theory of regularity structures gives a systematic description of such renormalisations of smooth stochastic PDEs (1.8) and defines a solution to (1.1) as their limits.

Conceptually, the aim of the theory of regularity structures is to solve a locally subcritical equation of the type (1.1) in generalised Hölder spaces in which the role of monomials is played by some "abstract objects". Formally, the strategy of formulating and solving a problem consists of three steps:

1. In the *algebraic* step, one builds the following objects:

    - A finite dimensional vector space $\mathcal{T}$ that allows to describe a kind of "Taylor expansions" of the solution around any point in space-time. The basis elements of $\mathcal{T}$ are some "abstract objects" and play the same role as the monomials in Taylor expansions, but, in contrast to the classical theory, they can correspond to quite general functions and/or distributions.
    - A group $\mathcal{G}$ of linear transformations of $\mathcal{T}$, whose applications usually correspond to changes of the localisation point in a classical Taylor expansion.
    - A finite dimensional renormalisation group $\mathfrak{R}$, which describes transformations of $\mathcal{T}$ corresponding to the renormalisation procedure mentioned above.

2. In the *analytical* step, one defines the following objects:

    - A *model* which is a pair of operators (Π, Γ) such that, for each space-time point $z \in \mathbb{R}^{d+1}$, the map $\Pi_z : \mathcal{T} \to \mathcal{S}'(\mathbb{R}^{d+1})$ transforms "abstract Taylor expansions" into localised in space-time concrete functions or distributions. Furthermore, for each $z, \bar{z} \in \mathbb{R}^{d+1}$, the operator $\Gamma_{z,\bar{z}} \in \mathcal{G}$ describes analytically the effect of changing the localisation point.
    - "Abstract Hölder spaces" $\mathcal{D}^\gamma$ (usually called spaces of *modelled distributions*) which correspond to the Hölder spaces with the Hölder exponent γ in the standard theory.
    - One formulates the mild version of (1.1) as a fixed point problem in one of the spaces $\mathcal{D}^\gamma$ (usually in a weighted version of $\mathcal{D}^\gamma$ describing a blow-up at $t = 0$ coming from the initial condition in (1.1)), and one builds an "abstract" solution map for (1.1) by solving this fixed point problem.
    - One defines a "*reconstruction map*" which transforms "abstract solutions" to concrete functions or distributions.

3. In the *probabilistic* step, one builds a concrete model, described in the previous step, corresponding to the noise ξ driving the equation (1.1). On this step one choses precise values of renormalisation constants in order to defined a finite number of products that have no classical meanings.

1.3 Results presented in thesis

The present thesis consists of two parts in which we use the theories of rough paths and regularity structures to investigate certain convergence properties of a large class of approximations of parabolic rough stochastic PDEs.

In the first part of the work, which contains Chapter 2, we consider a large class of approximations of the Burgers-type equations of the form (1.6), and we prove that the optimal rate of convergence in (1.7) is arbitrarily close to $\frac{1}{2}$. Formally, the main result of this part can be stated as follows.

Theorem 1.3.1. *Let for every* $\alpha \in \left(0, \frac{1}{2}\right)$ *the initial values in* (1.3) *and* (1.6) *satisfy*

$$\mathbb{E}\|u^0\|_{\mathcal{C}^\alpha} < \infty\,, \qquad \sup_{0<\varepsilon\leq 1} \mathbb{E}\|u^0_\varepsilon\|_{\mathcal{C}^\alpha} < \infty\,,$$

and let, for any $\alpha \in (0, \frac{1}{2})$*, there be a constant* C *independent of* ε *such that*

$$\mathbb{E}\|u^0 - u^0_\varepsilon\|_{\mathcal{C}^\alpha} \leq C\varepsilon^{\frac{1}{2}-\alpha}\,.$$

Then under quite general assumptions on the approximations in (1.6) *there exists a local solution* $\bar{u}$ *of a modified equation* (1.3) *defined on a random time interval* $[0, T^*)$*, and there exists a family of stopping times* T_ε *satisfying* $\lim_{\varepsilon\to 0} T_\varepsilon = T^*$ *in probability such that one has*

$$\lim_{\varepsilon\to 0} \mathbb{P}\left[\sup_{t\in[0,T_\varepsilon]} \|(\bar{u} - u_\varepsilon)(t)\|_{L^\infty} \geq \varepsilon^{\frac{1}{2}-\alpha}\right] = 0\,, \tag{1.11}$$

for every $\alpha > 0$*. The function* $\bar{u}$ *satisfies the equation* (1.3) *with the reaction term replaced by*

$$\bar{F}_i = F_i - \Lambda\,\mathrm{div}\,G_i\,,$$

where $i = 1, \ldots, n$*,* F_i *is the* i*-th element of the vector-valued function* F*,* G_i *is the* i*-th row of the matrix-valued function* G*, the constant* Λ *depends on the approximations in* (1.6) *and can be calculated explicitly.*

In order to get an optimal convergence rate, we need to consider convergence of the solutions in the Hölder spaces of the regularities close to zero. This approach creates difficulties when working with the rough integrals (1.5), since the classical theory of controlled rough paths was designed for Hölder spaces $\mathcal{C}^\alpha$ with $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$, and the bounds on the rough integrals obtained in [HMW14] hold only in these spaces. To have reasonable bounds in the Hölder spaces of lower regularity, we have to include into the definition of the rough integrals the iterated integrals of higher order of the controlling process X defined in (1.4). In [HMW14] it was enough to consider only the iterated integrals of order two. In particular, the smaller α is in (1.11), the more iterated integrals we have to consider to define the rough integral (1.5).

This result has been published in *Stochastic Partial Differential Equations: Analysis and Computations* [HM15b].

In the second part of this thesis, which contains Chapters 3 and 4, we develop a general framework for spatial discretisations of locally subcritical parabolic stochastic PDEs of the form (1.1) whose solutions are provided by the theory of regularity structures and which are functions in the time variable. As a particular example, we consider in Chapter 5 spatial discretisations of the dynamical Φ^4_3 model (1.9) on the dyadic grid $\mathbb{T}^3_\varepsilon \subset \mathbb{T}^3$ with the mesh size $\varepsilon > 0$ of the form

$$\frac{d}{dt}\Phi^\varepsilon = \Delta^\varepsilon \Phi^\varepsilon + C^{(\varepsilon)}\Phi^\varepsilon - \big(\Phi^\varepsilon\big)^3 + \xi^\varepsilon\;, \qquad \Phi^\varepsilon(0,\cdot) = \Phi^\varepsilon_0(\cdot)\;, \tag{1.12}$$

where Δ^ε is the nearest-neighbor approximation of the Laplacian Δ, Φ^ε_0 is some periodic initial value and the discretisation of the noise ξ is given by

$$\xi^\varepsilon(t,x) \stackrel{\text{def}}{=} \varepsilon^{-3}\langle \xi(t,\cdot), \mathbf{1}_{|\cdot - x|_\infty \leq \varepsilon/2}\rangle\;, \qquad (t,x) \in \mathbb{R} \times \mathbb{T}^3_\varepsilon\;,$$

where $|\cdot|_\infty$ is the supremum norm in $\mathbb{R}^3$. Our result concerning the discrete dynamical Φ^4_3 model can be formulated as follows (one can find the precise definitions of the discrete analogues of the Hölder norms in Section 4.1.1).

Theorem 1.3.2. *In the described settings, let $\Phi_0 \in \mathcal{C}^\eta(\mathbb{R}^3)$ almost surely, for some $\eta > -\frac{2}{3}$, let Φ be the unique maximal solution of* (1.10) *on a random time interval $[0, T_\star)$, and let Φ^ε be the unique global solution of* (1.12). *If the initial data satisfies*

$$\lim_{\varepsilon \to 0} \|\Phi_0; \Phi^\varepsilon_0\|^{(\varepsilon)}_{\mathcal{C}^\eta} = 0$$

almost surely, then for every $\alpha < -\frac{1}{2}$ there is a sequence of renormalisation constants $C^{(\varepsilon)} \sim \varepsilon^{-1}$ in (1.12) *and a sequence of stopping times T_ε satisfying $\lim_{\varepsilon\to 0} T_\varepsilon = T_\star$ in probability such that, for every $\bar\eta < \eta \wedge \alpha$, and for any $\delta > 0$ small enough, one has the limit in probability*

$$\lim_{\varepsilon \to 0} \|\Phi; \Phi^\varepsilon\|^{(\varepsilon)}_{\mathcal{C}^{\delta,\alpha}_{\bar\eta, T_\varepsilon}} = 0\;.$$

Our main motivation to prove this convergence result goes back to the seminal article [BFS83], where the authors prove that lattice approximations μ_ε to the Φ^4_3

measure are tight as the mesh size ε goes to 0. These measures are given by

$$\mu_\varepsilon(\Phi^\varepsilon) \stackrel{\text{def}}{=} e^{-S_\varepsilon(\Phi^\varepsilon)} \prod_{x\in\mathbb{T}_\varepsilon^3} d\Phi^\varepsilon(x)/Z_\varepsilon \;,$$

where Φ^ε is any function on $\mathbb{T}_\varepsilon^3$, Z_ε is a normalisation factor, called "*partition function*", and the "*action*" S_ε is defined by

$$S_\varepsilon(\Phi^\varepsilon) \stackrel{\text{def}}{=} \frac{\varepsilon}{2}\sum_{x\sim y}\big(\Phi^\varepsilon(x)-\Phi^\varepsilon(y)\big)^2 - \frac{C^{(\varepsilon)}\varepsilon^3}{2}\sum_{x\in\mathbb{T}_\varepsilon^3}\Phi^\varepsilon(x)^2 + \frac{\varepsilon^3}{4}\sum_{x\in\mathbb{T}_\varepsilon^3}\Phi^\varepsilon(x)^4 \;, \quad (1.13)$$

with the first sum running over all the nearest neighbours on the grid, when each pair x, y is counted twice. Since these measures are invariant for the finite difference approximations (1.12), showing that these converge to (1.10) straightforwardly implies that any accumulation point of μ_ε is invariant for the solutions of (1.10). These accumulation points are known to coincide with the Φ^4_3 measure μ, see [Par77], thus showing that μ is indeed invariant for (1.10), as one might expect. Heuristically, the measure μ can be written as

$$\mu(\Phi) \sim e^{-S(\Phi)} \prod_{x\in\mathbb{T}^3} d\Phi(x) \;,$$

for every $\Phi \in \mathcal{S}'(\mathbb{R}^3)$. In this case the "action" S is a limit of its finite difference approximations (1.13), i.e. it is formally given by

$$S(\Phi) = \int_{\mathbb{T}^3}\left(\frac{1}{2}\big(\nabla\Phi(x)\big)^2 - \frac{\infty}{2}\Phi(x)^2 + \frac{1}{4}\Phi(x)^4\right)dx \;.$$

With this notation at hand, an important corollary of Theorem 1.3.2 is the following result.

Corollary 1.3.3. *In the described context, for μ-almost every initial condition Φ_0, the solution of* (1.10) *constructed in [Hai14] is almost surely global in time. In particular, this yields a reversible Markov process on $\mathcal{C}^\alpha\big(\mathbb{R}^3\big)$, with α as in Theorem 1.3.2, for which the Φ^4_3 measure is invariant.*

These results have been presented in the preprint [HM15a] and have been submitted for publication to the *Annals of Probability*.

Since our framework is not designed specifically for the Φ^4_3 equation, it lays the foundations of a systematic approximation theory which can in principle be applied to many other singular stochastic PDEs, e.g. stochastic Burgers-type equations [Hai11, HMW14, HM15b], the KPZ equation [KPZ86, BG97, Hai13], or the continuous parabolic Anderson model [Hai14, HL15].

1.4 Outline of thesis

In Chapter 2 of this thesis we provide basics of the rough paths theory and the theory of controlled rough paths. Furthermore, we define a notion of solution for the rough stochastic Burgers-type equation (1.3) and obtain the optimal rate of convergence in the uniform topology of their natural approximations (1.6). In the other chapters of the thesis we work with the theory of regularity structures. In particular, in Chapter 3 we define the principal objects of the theory and provide their fundamental properties. Moreover, in this chapter we give a notion of solution for a large class of locally subcritical equations of the form (1.1). In Chapter 4 we develop a modification of the theory of regularity structures, which allows to reformulate spatially discretised rough stochastic PDEs on the "abstract" level. Finally, in Chapter 5 we apply this theory to analysis of discretisations of the Φ^4_3 equation (1.10).

Chapter 2

Approximations of rough stochastic Burgers-type equations

2.1 Introduction

The goal of this chapter is to study numerical approximations of stochastic PDEs of Burgers type on the circle $\mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}/(2\pi\mathbb{Z})$ given by

$$du = \Big(\nu\Delta u + F(u) + G(u)\,\partial_x u\Big)dt + \sigma dW(t)\;, \qquad u(0) = u^0\;. \tag{2.1}$$

Here, the solution $u : \mathbb{R}_+ \times \mathbb{T} \times \Omega \to \mathbb{R}^n$ is given on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\Delta \stackrel{\text{def}}{=} \partial_x^2$ is the Laplace operator on the circle $\mathbb{T}$, the derivative ∂_x is understood in the sense of distributions, the function $F : \mathbb{R}^n \to \mathbb{R}^n$ is of class $\mathcal{C}^1$, the function $G : \mathbb{R}^n \to \mathbb{R}^{n\times n}$ is of class $\mathcal{C}^\infty$, and $\nu, \sigma \in \mathbb{R}_+$ are some positive constants. Finally, W is an L^2-cylindrical Wiener process [DPZ02], i.e. equation (2.1) is driven by space-time white noise. The product appearing in the term $G(u)\,\partial_x u$ is matrix-vector multiplication.

The difficulty in dealing with (2.1) comes from the nonlinearity $G(u)\,\partial_x u$ and is caused by the low space-time regularity of the driving noise. Indeed, it is well-known that the pairing

$$\mathcal{C}^\alpha \times \mathcal{C}^\beta \ni (v, u) \mapsto v\,\partial_x u$$

is well defined if and only if $\alpha + \beta > 1$, see Lemma 2.1.1. On the other hand, one expects solutions of (2.1) to have the spatial regularity of the solution of the linearised equation

$$dX(t) = \nu\Delta X dt + \sigma dW(t)\;. \tag{2.2}$$

For any fixed time $t > 0$, the solution of the stochastic heat equation (2.2) has almost surely Hölder regularity $\alpha < \frac{1}{2}$, but is *not* $\frac{1}{2}$-Hölder continuous, see [Wal86, DPZ02, Hai09]. This implies in particular that the product $G(X)\,\partial_x X$ is not well-defined in this case, and it is not a priori clear how to define a solution of the equation (2.1).

In the case $G \equiv 0$ this problem does of course not occur. Equations of this type and their numerical approximations were well studied and the results can be found in [Gyö98b, Gyö99]. Moreover, it was shown in [DG01] that the optimal rate of uniform convergence in this case is $\frac{1}{2} - \kappa$, for every $\kappa > 0$, as the mesh size of the spatial discretisation tends to zero.

For non-zero G, the difficulty can be easily overcome in the gradient case, i.e. when $G = \nabla\mathcal{G}$ for some sufficiently regular function $\mathcal{G} : \mathbb{R}^n \to \mathbb{R}^n$. In this case, postulating the chain rule, the nonlinear term can be rewritten as

$$G(u(t,x))\,\partial_x u(t,x) = \partial_x \mathcal{G}(u(t,x)) \;, \tag{2.3}$$

which is a well-defined distribution as soon as u is continuous. The existence and uniqueness results in the gradient case can be found in [Gyö98a, DPDT94]. In the article [AG06], the finite difference scheme was studied for the case $G(u) = u$, and L^2-convergence was shown with rate $\frac{1}{2} - \kappa$, for every $\kappa > 0$. The same rate of convergence was obtained in [BJ13] in the L^∞-topology for Galerkin approximations.

For a general sufficiently smooth function G, a notion of solution for (2.1) was introduced in [Hai11]. The key idea of the approach was to test the nonlinearity with a smooth test function φ and to formally rewrite it as

$$\int_{-\pi}^{\pi} \varphi(x) G(u(t,x))\,\partial_x u(t,x)\,dx = \int_{-\pi}^{\pi} \varphi(x) G(u(t,x))\,d_x u(t,x)\;. \tag{2.4}$$

As it was stated above, we expect u to behave locally like the solution to the linearised equation (2.2). It was shown in [Hai11] that the latter can be viewed in a canonical way as a process with values in a space of rough paths. This correctly suggests that the theory of controlled rough paths [Gub04, Gub10] could be used to deal with the integral (2.4) in the pathwise sense. The quantity (2.4) is uniquely defined up to a choice of the iterated integral which represents the integral of u with respect to itself. This implies that for different choices of the iterated integral we obtain different solutions, which is similar to the choice between Itô and Stratonovich stochastic integrals in the theory of SDEs. In the present situation however, there is a unique choice for the iterated integral which respects the symmetry of the linearised equation under the substitution $x \mapsto -x$, and this corresponds to the "Stratonovich solution". This natural choice is also the one for which the chain rule (2.3) holds in the particular case when G is a gradient.

Using the rough path approach, numerical approximations to (2.1) in the gradient case without using the chain rule were studied in [HM12]. It was shown that the corresponding approximate solutions converge in suitable Sobolev spaces to a limit which solves (2.1) with an additional correction term, which can be computed explicitly. This term is an analogue to the Itô-Stratonovich correction term in the classical theory of SDEs.

In [HW13], the solution theory was extended to Burgers-type equations with multiplicative noise (i.e. when the multiplier of the noise term is a nonlinear local function $\theta(u)$ of the solution). Analysis of numerical schemes approximating the equation in the multiplicative case was performed in [HMW14], where the appearance of a correction term was observed and the rate of convergence in the uniform topology was shown to be of order $\frac{1}{6} - \kappa$, for every $\kappa > 0$.

In this chapter, we prove that in the case of additive noise the rate of convergence in the supremum norm is $\frac{1}{2} - \kappa$, for every $\kappa > 0$, see Theorem 2.2.7 below. Actually, it turns out to be technically advantageous to consider convergence in Hölder spaces with Hölder exponent very close to zero. The main difference to [HMW14] is that we cannot use the classical theory of controlled rough paths which applies only in the Hölder spaces of regularity from $\left(\frac{1}{3}, \frac{1}{2}\right]$, to approximate the rough integral (2.4). To show the convergence in the Hölder spaces of lower regularity, we use the results from [Gub10], which generalize the theory of controlled rough paths for functions of any positive regularity.

Structure of the chapter

This chapter is structured in the following way. In Section 2.2 we formulate the approximate equations and state the main result. In Section 2.3 we review the theories of rough paths and controlled rough paths. Section 2.4 is devoted to the results obtained in [Hai11]. In particular, here we provide a notion of solution and the existence and uniqueness results for the Burgers-type equations with additive noise. In Section 2.5 we define the rough integrals and formulate the mild solutions to the approximate equations in a way appropriate for working in the Hölder spaces of low regularity. Section 2.6 provides regularity properties of the heat semigroup and its approximate counterpart in the Hölder/Besov spaces. The following Section 2.7 gives bounds on the respective terms in the continuous and approximate equations. The proof of the convergence result, Theorem 2.2.7 below, is provided in Section 2.8.

2.1.1 Spaces, norms and notation

In this chapter we will use the following notation. For functions $X : \mathbb{R} \to \mathbb{R}^n$ (or $\mathbb{R}^{n\times n}$) and $R : \mathbb{R}^2 \to \mathbb{R}^n$ (or $\mathbb{R}^{n\times n}$), such that R vanishes on the diagonal, we define respectively Hölder seminorms with a given parameter $\alpha \in (0,1)$:

$$\|X\|_\alpha \stackrel{\text{def}}{=} \sup_{x\neq y} \frac{|X(x)-X(y)|}{|x-y|^\alpha}\;, \qquad \|R\|_\alpha \stackrel{\text{def}}{=} \sup_{x\neq y} \frac{|R(x,y)|}{|x-y|^\alpha}\;. \tag{2.5}$$

By $\mathcal{C}^\alpha$ and $\mathcal{B}^\alpha$ respectively we denote the spaces of functions for which these seminorms are finite. Then $\mathcal{C}^\alpha$ endowed with the norm $\|\cdot\|_{\mathcal{C}^\alpha} \stackrel{\text{def}}{=} \|\cdot\|_{L^\infty} + \|\cdot\|_\alpha$ is a Banach space. $\mathcal{B}^\alpha$ is a Banach space endowed with $\|\cdot\|_{\mathcal{B}^\alpha} \stackrel{\text{def}}{=} \|\cdot\|_\alpha$.

The Hölder space $\mathcal{C}^\alpha$ of regularity $\alpha \geq 1$ consists of $\lfloor\alpha\rfloor$ times continuously differentiable functions whose $\lfloor\alpha\rfloor$-th derivative is $(\alpha-\lfloor\alpha\rfloor)$-Hölder continuous. The space $\mathcal{C}^0$ consists of continuous functions and is equipped with the supremum norm.

For $\alpha < 0$ we define the space $\mathcal{C}^\alpha$ in the following way. Any distribution ψ defined on the circle $\mathbb{T}$ we write as the Fourier series

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} \widehat{\psi}(k)\, e^{ikx}\;,$$

where $\widehat{\psi}$ is the Fourier transform of ψ on the circle. For $n \geq 1$ we define the n-th Littlewood-Paley block of ψ as

$$\delta_n\psi(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \sum_{2^{n-1}\leq|k|<2^n} \widehat{\psi}(k)\, e^{ikx}\;,$$

and by definition $\delta_0\psi \equiv \widehat{\psi}(0)/\sqrt{2\pi}$. Then, for any $\alpha < 0$, the Besov space $\mathcal{B}^\alpha_{\infty,\infty}(\mathbb{T})$ consists of those distributions on $\mathbb{T}$, for which the norm

$$\|\psi\|_{\mathcal{B}^\alpha_{\infty,\infty}} \stackrel{\text{def}}{=} \sup_{n\geq 0} 2^{\alpha n}\|\delta_n\psi\|_{L^\infty}$$

is finite. We denote $\mathcal{C}^\alpha \stackrel{\text{def}}{=} \mathcal{B}^\alpha_{\infty,\infty}(\mathbb{T})$ and identify these distributions with their periodic extensions to $\mathbb{R}$. In the same way, we can define the Besov space $\mathcal{B}^\alpha_{\infty,\infty}(\mathbb{T})$ for $\alpha \geq 0$. Then for $\alpha \notin \mathbb{N}$ this space coincides with the Hölder space $\mathcal{C}^\alpha(\mathbb{T})$. The proof of this fact and more properties of the Besov spaces can be found in [BCD11]. One of the important properties, whose proof is provided in [BCD11, Thm. 2.85], concerns products of two functions/distributions from certain Besov spaces.

Lemma 2.1.1. *Let $\varphi \in \mathcal{C}^\alpha$ and $\psi \in \mathcal{C}^\beta$, where $\beta \leq \alpha$ and $\alpha+\beta > 0$. Then the* *product $\varphi\psi$ is well defined and there exists a constant C, depending on α and β, such that*

$$\|\varphi\psi\|_{\mathcal{C}^\beta} \le C\|\varphi\|_{\mathcal{C}^\alpha}\|\psi\|_{\mathcal{C}^\beta} .$$

We also define space-time Hölder norms, i.e. for some $T > 0$ and functions $X : [0,T] \times \mathbb{T} \to \mathbb{R}^n$ (or $\mathbb{R}^{n\times n}$) and $R : [0,T] \times \mathbb{T}^2 \to \mathbb{R}^n$ (or $\mathbb{R}^{n\times n}$) vanishing on the diagonal, any $\alpha \in \mathbb{R}$ and any $\beta > 0$ we define

$$\|X\|_{\mathcal{C}^\alpha_T} \stackrel{\text{def}}{=} \sup_{s\in[0,T]} \|X(s)\|_{\mathcal{C}^\alpha} , \qquad \|R\|_{\mathcal{B}^\beta_T} \stackrel{\text{def}}{=} \sup_{s\in[0,T]} \|R(s)\|_{\mathcal{B}^\beta} . \tag{2.6}$$

We denote by $\mathcal{C}^\alpha_T$ and $\mathcal{B}^\alpha_T$ respectively the spaces of functions/distributions for which the norms (2.6) are finite. Furthermore, in order to deal with functions X exhibiting a blow-up with rate $\eta > 0$ near $t = 0$, we define the norm

$$\|X\|_{\mathcal{C}^\alpha_{\eta,T}} \stackrel{\text{def}}{=} \sup_{s\in(0,T]} s^\eta \|X(s)\|_{\mathcal{C}^\alpha} .$$

Similarly to above, we denote by $\mathcal{C}^\alpha_{\eta,T}$ the space of functions/distributions for which this norm is finite.

By $\|\cdot\|_{\mathcal{C}^\alpha\to\mathcal{C}^\beta}$ we denote the operator norm of a linear map acting from the space $\mathcal{C}^\alpha$ to $\mathcal{C}^\beta$. When we write $x \lesssim y$, we mean that there is a constant C, independent of the relevant quantities, such that $x \le Cy$.

2.2 Approximate equations and a convergence result

As before we assume that $F \in \mathcal{C}^1$ and $G \in \mathcal{C}^\infty$ in (2.1). For $\varepsilon \in (0,1]$ we consider the approximate stochastic PDEs on the circle $\mathbb{T}$ given by

$$du_\varepsilon = \Big(\nu\Delta_\varepsilon u_\varepsilon + F(u_\varepsilon) + G(u_\varepsilon)D_\varepsilon u_\varepsilon\Big)dt + \sigma H_\varepsilon dW(t) , \qquad u_\varepsilon(0) = u^0_\varepsilon . \tag{2.7}$$

Here, the operators Δ_ε, D_ε and H_ε are defined as Fourier multipliers providing approximations of Δ, ∂_x and the identity operator respectively, and are given by

$$\widehat{\Delta_\varepsilon u}(k) = -k^2 \mathscr{m}(\varepsilon k)\widehat{u}(k), \quad \widehat{D_\varepsilon u}(k) = ik\mathscr{g}(\varepsilon k)\widehat{u}(k), \quad \widehat{H_\varepsilon W}(k) = \mathscr{h}(\varepsilon k)\widehat{W}(k),$$

where by $\widehat{u}$ we denote the Fourier transform of u on $\mathbb{T}$. Below we provide the assumptions on the functions $\mathfrak{m}$, $\mathfrak{g}$ and $\mathfrak{h}$. We start with the assumptions on $\mathfrak{m}$.

Assumption 2.2.1. *The function $\mathfrak{m} : \mathbb{R} \to (0, \infty]$ is even, satisfies $\mathfrak{m}(0) = 1$, is continuously differentiable on the interval $[-\delta, \delta]$ for some $\delta > 0$, and there exists a constant $c_{\mathfrak{m}} \in (0, 1)$ such that $\mathfrak{m} \geq c_{\mathfrak{m}}$. Furthermore, the functions $b_t : \mathbb{R} \to \mathbb{R}$, defined by*

$$b_t(x) \stackrel{def}{=} \exp(-x^2 \mathfrak{m}(x)t) ,$$

are uniformly bounded in $t > 0$ in the bounded variation norm, i.e.

$$\sup_{t>0} |b_t|_{\mathrm{BV}} < \infty .$$

Our next assumption concerns $\mathfrak{g}$, which defines the approximation to the spatial derivative.

Assumption 2.2.2. *There exists a signed Borel measure μ on $\mathbb{R}$ such that, for $k \in \mathbb{Z}$,*

$$\int_{\mathbb{R}} e^{ikx} \mu(dx) = ik\mathfrak{g}(k) ,$$

and such that, for any integer $k \geq 1$, one has

$$\mu(\mathbb{R}) = 0 , \qquad |\mu|(\mathbb{R}) < \infty , \qquad \int_{\mathbb{R}} x\, \mu(dx) = 1 , \qquad \int_{\mathbb{R}} |x|^k \, |\mu|(dx) < \infty .$$

In particular, the approximate derivative can be expressed as

$$(D_\varepsilon u)(x) = \frac{1}{\varepsilon} \int_{\mathbb{R}} u(x + \varepsilon y)\, \mu(dy) ,$$

where we identify $u : \mathbb{T} \to \mathbb{R}$ with its periodic extension to all $\mathbb{R}$. Our last assumption is on the function $\mathfrak{h}$, which defines the approximation of noise.

Assumption 2.2.3. *The function $\mathfrak{h}$ is even, bounded, and is such that the functions $\mathfrak{h}^2/\mathfrak{m}$ and $\mathfrak{h}/(\mathfrak{m}+1)$ are of bounded variation. Furthermore, $\mathfrak{h}$ is twice differentiable at the origin with $\mathfrak{h}(0) = 1$ and $\mathfrak{h}'(0) = 0$.*

The difference with the assumptions in [HMW14] is that we require in Assumption 2.2.2 all the moments of the measure μ to be finite and in Assumption 2.2.3

the function $\hbar/(m+1)$ to be of bounded variation. We use the latter assumption in Lemma 2.5.1 in order to use the bounds on lifted rough paths obtained in [FGGR16]. Before we proceed, we provide some examples of discretisations mentioned in [HM12] which satisfy our assumptions. One can also find in this article the precise values of the correction terms for these examples which are given in (2.10) in a general form.

Example 2.2.4 (No discretisation). We do not discretise the Laplacian and the noise at all, i.e. we take $m = \hbar = 1$, and we approximate the derivative operator by choosing the measure μ in Assumption 2.2.2 to be

$$\mu = \frac{\delta_a - \delta_{-b}}{a+b} \;, \tag{2.8}$$

for some constants $a, b \geq 0$ with $a + b > 0$.

Example 2.2.5 (Finite difference discretisation). In this case we consider a grid with the mesh size $\varepsilon = 2\pi/N$ for an odd integer N, and we define the finite difference Laplacian by

$$\big(\Delta_\varepsilon u\big)(x) = \frac{1}{\varepsilon^2}\Big(u(x+\varepsilon) - 2u(x) + u(x-\varepsilon)\Big) \;.$$

We also identify a function u with a trigonometric polynomial of degree $\frac{N-1}{2}$ agreeing with u at the grid with the mesh size ε. This corresponds to the choice $\hbar = \mathbf{1}_{[0,\pi)}$ and

$$m(x) = \begin{cases} \frac{4}{x^2}\sin^2\big(\frac{x}{2}\big) \;, & \text{for} \quad x \in [0,\pi) \;, \\ +\infty \;, & \text{for} \quad x \in [\pi, +\infty) \;. \end{cases}$$

Furthermore, we discretise the derivative by (2.8) with integer values a and b.

Example 2.2.6 (Galerkin discretisation). In this case, we approximate Δ and ξ by only keeping those Fourier modes that appear in the approximation by trigonometric polynomials. This corresponds to the choice (2.8), $\hbar = \mathbf{1}_{[0,\pi)}$ and

$$m(x) = \begin{cases} 1 \;, & \text{for} \quad x \in [0,\pi) \;, \\ +\infty \;, & \text{for} \quad x \in [\pi, +\infty) \;. \end{cases}$$

As it was mentioned above, one can expect to obtain a correction term in (2.1), when taking a limit of (2.7). Thus, we denote by $\bar{u}$ the solution of the modified equation

$$d\bar{u} = \Big(\nu\Delta\bar{u} + \bar{F}(\bar{u}) + G(\bar{u})\,\partial_x\bar{u}\Big)dt + \sigma dW(t)\;, \qquad \bar{u}(0) = u^0\;, \tag{2.9}$$

where, for $i = 1, \ldots, n$, the modified reaction term is given by

$$\bar{F}_i \stackrel{\text{def}}{=} F_i - \Lambda \operatorname{div} G_i\;.$$

Here, we denote by F_i the i-th element of the vector-valued function F, and by G_i the i-th row of the matrix-valued function G, and the correction constant is defined by

$$\Lambda \stackrel{\text{def}}{=} \frac{\sigma^2}{2\pi\nu}\int_{\mathbb{R}_+}\int_{\mathbb{R}} \frac{(1-\cos(yt))\,\mathscr{h}^2(t)}{t^2\,\mathscr{m}(t)}\,\mu(dy)\,dt\;. \tag{2.10}$$

It follows from our assumptions that Λ is well-defined. In fact, Assumption 2.2.3 says that $|\mathscr{h}^2/\mathscr{m}|$ is bounded, and by Assumption 2.2.2 the measure μ has a finite second moment, what yields existence of Λ.

As we do not assume boundedness of the functions $\bar{F}$ and G, and their derivatives, the solution $\bar{u}$ can in principle blow up in a finite time. To overcome this difficulty we consider solutions only up to some stopping times. More precisely, for any $K > 0$ we define the stopping time

$$T_K^* \stackrel{\text{def}}{=} \inf\big\{t > 0 : \|\bar{u}(t)\|_{\mathcal{C}^0} \geq K\big\}\;. \tag{2.11}$$

The blow-up time of $\bar{u}$ is then defined as the limit $T^* \stackrel{\text{def}}{=} \lim_{K\uparrow\infty} T_K^*$ in probability.

Our main theorem in this chapter gives the convergence rate of the solutions of the approximate equations (2.7) to the solution of the modified equation (2.9).

Theorem 2.2.7. *Let for every $\eta \in \big(0, \frac{1}{2}\big)$ the initial values satisfy*

$$\mathbb{E}\|u^0\|_{\mathcal{C}^\eta} < \infty\;, \qquad \sup_{0<\varepsilon\leq 1}\mathbb{E}\|u_\varepsilon^0\|_{\mathcal{C}^\eta} < \infty\;.$$

Then, there exists $\alpha_0 > 0$ such that if, for some $\alpha \in (0, \alpha_0]$ and some constant $C > 0$

independent of ε, one has

$$\mathbb{E}\|u^0 - u_\varepsilon^0\|_{\mathcal{C}^\alpha} \leq C\varepsilon^{\frac{1}{2}-\alpha}\,,$$

then there exists a family of stopping times T_ε satisfying $\lim_{\varepsilon\to 0} T_\varepsilon = T^*$ in probability such that

$$\lim_{\varepsilon\to 0}\mathbb{P}\Big[\|\bar{u} - u_\varepsilon\|_{\mathcal{C}^0_{T_\varepsilon}} \geq \varepsilon^{\frac{1}{2}-\alpha}\Big] = 0\,.$$

Remark 2.2.8. The rate of convergence obtained in [HMW14] is "almost" $\frac{1}{6}$, in the sense that it is $\frac{1}{6} - \kappa$, for any $\kappa > 0$. To improve this result we consider convergence of the solutions in the Hölder spaces of the regularities close to zero. This approach creates difficulties when working with the rough integrals (2.4). In fact, the bounds on the rough integrals, in particular in [HMW14, Lem. 5.3], hold only in the Hölder spaces $\mathcal{C}^\alpha$ with $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$ and the norms explode as α approaches $\frac{1}{3}$. To have reasonable bounds in the Hölder spaces of lower regularity, we have to include into the definition of the rough integrals the iterated integrals of the controlling process X of higher order. In [HMW14] it was enough to consider only the iterated integrals of order two. In particular, the smaller α is in Theorem 2.2.7, the more iterated integrals we have to consider to define the rough integral (2.4), see Section 2.3 for more details.

Remark 2.2.9. If the function G is only of class $\mathcal{C}^p$ for some integer $p \geq 3$, we can consider the iterated integrals of X only up to the order $p-1$, see Section 2.5.1. As a consequence, the argument in the proof of Theorem 2.2.7 gives the rate of convergence only "almost" $\frac{1}{2} - \frac{1}{p}$. This is precisely the rate of convergence obtained in [HMW14], where p was taken to be 3.

Remark 2.2.10. If the functions $\bar{F}$ and G are bounded together with all of their derivatives, the solution $\bar{u}$ is global, i.e. $T^* = +\infty$ a.s., see [Hai11, Thm. 3.6]. In this case, the argument of the proof of Theorem 2.2.7 shows that one can take for example $T_\varepsilon = T$ a.s. for any fixed time $T > 0$. Since it is straightforward in this case to obtain uniform bounds (on finite time intervals) on the p-th moment of the solution for every p, uniformly over $\varepsilon \in (0, 1]$, this implies strong L^p-convergence of the approximate solutions on every bounded time interval. In general, we do expect to have $T^* < \infty$, which of course precludes any form of L^p-convergence without cutoffs.

Remark 2.2.11. By changing the time variable and the functions in (2.1) by constant multipliers, we can obtain an equivalent equation with $\nu = 1$. Moreover, we can assume for simplicity $\sigma = 1$. In what follows we only consider these values of the constants.

2.3 Elements of rough paths theory

In this section we provide an overview of rough paths theory and controlled rough paths. For more information on rough paths theory we refer to the original article [Lyo98] and to the monographs [LQ02, LCL07, FV10, FH14].

One of the aims of rough paths theory is to provide a consistent and robust way of defining the integral

$$\int_s^t Y(r) \otimes dX(r) \,, \tag{2.12}$$

for processes $Y, X \in \mathcal{C}^\alpha$ with any Hölder exponent $\alpha \in \left(0, \frac{1}{2}\right]$. If $\alpha > \frac{1}{2}$, then the integral can be defined in Young's sense [You36] as the limit of Riemann sums. If $\alpha \leq \frac{1}{2}$, however, the Riemann sums may diverge (or fail to converge to a limit independent of the partition) and the integral cannot be defined in this way. Given $X \in \mathcal{C}^\alpha$ with $\alpha \in \left(0, \frac{1}{2}\right]$, the theory of (controlled) rough paths allows to define (2.12) in a consistent way for a certain class of integrands Y. To this end however, one has to consider not only the processes X and Y, but suitable additional "higher order" information.

We fix $\alpha \in \left(0, \frac{1}{2}\right]$ and $p = \lfloor 1/\alpha \rfloor$ to be the largest integer such that $p\alpha \leq 1$. We then define the p-step truncated tensor algebra

$$T^{(p)}(\mathbb{R}^n) \stackrel{\text{def}}{=} \bigoplus_{k=0}^{p} (\mathbb{R}^n)^{\otimes k} \,,$$

whose basis elements can be labelled by words of length not exceeding p (including the empty word $\varnothing$), based on the alphabet $\mathcal{A} \stackrel{\text{def}}{=} \{1, \ldots, n\}$. We denote this set of words by $\mathcal{A}_p$. Then the correspondence $\mathcal{A}_p \to T^{(p)}(\mathbb{R}^n)$ is given by $w \mapsto e_w$ with $e_w \stackrel{\text{def}}{=} e_{w_1} \otimes \ldots \otimes e_{w_k}$, for a word $w = w_1 \ldots w_k$ and $e_\varnothing \stackrel{\text{def}}{=} 1 \in (\mathbb{R}^n)^{\otimes 0} \approx \mathbb{R}$, where $\{e_i\}_{i \in \mathcal{A}}$ is the canonical basis of $\mathbb{R}^n$. We extend e_w to $T^{(p)}(\mathbb{R}^n)$ by linearity.

There is an operation $\sqcup\!\sqcup$, called *shuffle product* [Reu93], defined on the free algebra generated by $\mathcal{A}$. For any two words the shuffle product gives all the possible ways of interleaving them in the ways that preserve the original order of the letters. For example, if a, b and c are letters from $\mathcal{A}$, then one has the identity

$$ab \sqcup\!\sqcup ac = abac + 2aabc + 2aacb + acab \; .$$

We also define both the shuffle and the concatenation products of two elements from $T^{(p)}(\mathbb{R}^n)$, i.e. for any two words $w, \bar{w} \in \mathcal{A}_p$ we define

$$e_w \sqcup\!\sqcup e_{\bar{w}} \stackrel{\text{def}}{=} e_{w \sqcup\!\sqcup \bar{w}} \; , \qquad e_w \otimes e_{\bar{w}} \stackrel{\text{def}}{=} e_{w\bar{w}} \; ,$$

if the sums of the lengths of the two words do not exceed p and

$$e_w \sqcup\!\sqcup e_{\bar{w}} \stackrel{\text{def}}{=} e_w \otimes e_{\bar{w}} \stackrel{\text{def}}{=} 0$$

otherwise. This is extended to all of $T^{(p)}(\mathbb{R}^n)$ by linearity. With these notations at hand, we give the following definition:

Definition 2.3.1. For $\alpha \in \left(0, \frac{1}{2}\right]$, a map $\mathbf{X} : \mathbb{R}^2 \to T^{(p)}(\mathbb{R}^n)$ is a *geometric rough path* of regularity α, where as above $p = \lfloor 1/\alpha \rfloor$, if

1. $\langle \mathbf{X}(s,t), e_w \sqcup\!\sqcup e_{\bar{w}} \rangle = \langle \mathbf{X}(s,t), e_w \rangle \langle \mathbf{X}(s,t), e_{\bar{w}} \rangle$, for any words $w, \bar{w} \in \mathcal{A}_p$ of respective lengths $|w|$ and $|\bar{w}|$ with $|w| + |\bar{w}| \leq p$,
2. $\mathbf{X}(s,t) = \mathbf{X}(s,u) \otimes \mathbf{X}(u,t)$, for any $s, u, t \in \mathbb{R}$,
3. $\big\| \langle \mathbf{X}, e_w \rangle \big\|_{\mathcal{B}^{\alpha|w|}} < \infty$, for any word $w \in \mathcal{A}_p$ of length $|w|$.

Given an α-regular rough path $\mathbf{X}$, we define the following quantity

$$\|\!|\mathbf{X}|\!\|_\alpha \stackrel{\text{def}}{=} \sum_{w \in \mathcal{A}_p \setminus \{\emptyset\}} \big\| \langle \mathbf{X}, e_w \rangle \big\|_{\mathcal{B}^{\alpha|w|}} \; . \tag{2.13}$$

If we define $X^i(t) \stackrel{\text{def}}{=} \langle \mathbf{X}(0,t), e_i \rangle$ for any $i \in \mathcal{A}$, then the components of $\mathbf{X}(s,t)$ of higher orders should be thought of as defining the iterated integrals

$$\langle \mathbf{X}(s,t), e_w \rangle \stackrel{\text{def}}{=} \int_s^t \ldots \int_s^{r_2} dX^{w_1}(r_1) \ldots dX^{w_k}(r_k) \; , \tag{2.14}$$

for $w = w_1 \dots w_k \in \mathcal{A}_p$. Of course, the integrals on the right hand side of (2.14) are not defined, as mentioned at the beginning of this chapter. Hence, for a given rough path $\mathbf{X}$, then the left hand side of (2.14) is the definition of the right hand side.

The conditions in Definition 2.3.1 ensure that the quantities (2.14) behave like iterated integrals. In particular, if X is a smooth function and we define $\mathbf{X}$ by (2.14) in Young's sense, then $\mathbf{X}$ satisfies the conditions of Definition 2.3.1, as was shown in [Che54]. In particular, if $x = e_i$ and $y = e_j$, for any two letters $i, j \in \mathcal{A}$, then the first property gives

$$\langle \mathbf{X}(s,t), e_i \otimes e_j \rangle + \langle \mathbf{X}(s,t), e_j \otimes e_i \rangle = X^i(s,t) X^j(s,t) \;,$$

where we write $X^i(s,t) \stackrel{\text{def}}{=} X^i(t) - X^i(s)$. This is the usual integration by parts formula. The second condition of Definition 2.3.1 provides the additivity property of the integral over consecutive intervals.

2.3.1 Controlled rough paths

The theory of controlled rough paths was introduced in [Gub04] for geometric rough paths of Hölder regularity from $\left(\frac{1}{3}, \frac{1}{2}\right]$. In [Gub10], the theory was generalised to rough paths (also non-geometric) of arbitrary positive regularity.

Definition 2.3.2. For $\alpha \in \left(0, \frac{1}{2}\right]$, $p = \lfloor 1/\alpha \rfloor$, a geometric rough path $\mathbf{X}$ of regularity α, and a function $Y : \mathbb{R} \to \left(T^{(p-1)}(\mathbb{R}^n)\right)^*$ (the dual of the truncated tensor algebra), we say that Y is *controlled* by $\mathbf{X}$ if, for every word $w \in \mathcal{A}_{p-1}$, one has the bound

$$\left| \langle Y(t), e_w \rangle - \langle Y(s), \mathbf{X}(s,t) \otimes e_w \rangle \right| \le C |t-s|^{(p-|w|)\alpha} \;,$$

for some constant $C > 0$.

An alternative statement of Definition 2.3.2 is that for every word $w \in \mathcal{A}_{p-1}$ there exists a function $R_Y^w \in \mathcal{B}^{(p-|w|)\alpha}$ such that

$$\langle Y(t), e_w \rangle = \sum_{\bar{w} \in \mathcal{A}_{p-|w|-1}} \langle Y(s), e_{\bar{w}} \otimes e_w \rangle \langle \mathbf{X}(s,t), e_{\bar{w}} \rangle + R_Y^w(s,t) \;. \qquad (2.15)$$

Given an α-regular geometric rough path $\mathbf{X}$, we then endow the space of all controlled paths Y with the seminorm

$$\|Y\|_{\mathcal{C}^\alpha_{\mathbf{X}}} \stackrel{\text{def}}{=} \sum_{w \in \mathcal{A}_{p-1}} \big\|\langle Y, e_w\rangle\big\|_{\mathcal{C}^\alpha} + \sum_{w \in \mathcal{A}_{p-2}} \big\|R^w_Y\big\|_{\mathcal{B}^{(p-|w|)\alpha}} \;. \tag{2.16}$$

For a rough path Y controlled by $\mathbf{X}$, one can define the integral (2.12) by

$$\fint_s^t Y(r)\, dX^i(r) \stackrel{\text{def}}{=} \lim_{|\mathcal{P}| \to 0} \sum_{[u,v] \in \mathcal{P}} \Xi_i(u,v) \;, \tag{2.17}$$

where we denoted as before $X^i(t) \stackrel{\text{def}}{=} \langle \mathbf{X}(0,t), e_i\rangle$ for $i \in \mathcal{A}$, and where

$$\Xi_i(u,v) \stackrel{\text{def}}{=} \sum_{w \in \mathcal{A}_{p-1}} \langle Y(u), e_w\rangle \langle \mathbf{X}(u,v), e_w \otimes e_i\rangle \;. \tag{2.18}$$

Here, the limit is taken over a sequence of partitions $\mathcal{P}$ of the interval $[s,t]$, whose diameters $|\mathcal{P}|$ tend to 0. It was proved in [Gub10, Thm. 8.5] that the rough integral (2.17) is well defined, i.e. the limit in (2.17) exists and is independent of the choice of partitions $\mathcal{P}$.

If every coordinate Y^j of the process Y is controlled by $\mathbf{X}$, then we denote the rough integral of Y with respect to X by

$$\left(\fint_s^t Y(r) \otimes dX(r)\right)_{ij} \stackrel{\text{def}}{=} \fint_s^t Y^j(r)\, dX^i(r) \;.$$

We use the symbol $\fint$ for the rough integral in (2.17), in order to remind the abuse of notation, since the integral depends not only on X^i and Y^j, but on much more information contained in $\mathbf{X}$ and Y. In the following proposition we provide several bounds on the rough integrals.

Proposition 2.3.3. *Let Y be controlled by a geometric rough path $\mathbf{X}$ of regularity $\alpha \in \big(0, \frac{1}{2}\big]$. Then there is a constant C, independent of Y and $\mathbf{X}$, such that*

$$\left|\fint_s^t Y(r) \otimes dX(r) - \Xi(s,t)\right| \leq C \|\!|\mathbf{X}|\!\|_\alpha \big\|Y\big\|_{\mathcal{C}^\alpha_{\mathbf{X}}} |t-s|^{\alpha(p+1)} \;, \tag{2.19}$$

$$\left\|\fint_s^{\cdot} Y(r) \otimes dX(r)\right\|_\alpha \leq C \|\!|\mathbf{X}|\!\|_\alpha \big\|Y\big\|_{\mathcal{C}^\alpha_{\mathbf{X}}} \;, \tag{2.20}$$

where we have used the seminorms defined in (2.5)*,* (2.13) *and* (2.16)*.*

Moreover, if $\bar{Y}$ is controlled by another rough path $\bar{\mathbf{X}}$ of regularity α, then there is a constant C, independent of $\mathbf{X}$, $\bar{\mathbf{X}}$, Y and $\bar{Y}$, such that

$$\begin{aligned}\Big\|\fint_s^\cdot Y(r)\otimes dX(r) - \fint_s^\cdot \bar{Y}(r)\otimes d\bar{X}(r)\Big\|_\alpha \qquad\qquad (2.21)\\ \le C\|\mathbf{X}-\bar{\mathbf{X}}\|_\alpha\Big(\|Y\|_{\mathcal{C}^\alpha_{\mathbf{X}}} + \|\bar{Y}\|_{\mathcal{C}^\alpha_{\bar{\mathbf{X}}}}\Big) + C\Big(\|\mathbf{X}\|_\alpha + \|\bar{\mathbf{X}}\|_\alpha\Big)\|Y,\bar{Y}\|_{\mathcal{C}^\alpha_{\mathbf{X},\bar{\mathbf{X}}}}\;,\end{aligned}$$

where we have used the quantity

$$\|Y,\bar{Y}\|_{\mathcal{C}^\alpha_{\mathbf{X},\bar{\mathbf{X}}}} \stackrel{\text{def}}{=} \sum_{w\in\mathcal{A}_{p-1}} \big\|\langle Y-\bar{Y}, e_w\rangle\big\|_{\mathcal{C}^\alpha} + \sum_{w\in\mathcal{A}_{p-2}} \big\|R_Y^w - R_{\bar{Y}}^w\big\|_{\mathcal{B}^{(p-|w|)\alpha}}\;,$$

involving the terms of the controlled rough path from (2.15).

Proof. These bounds follow from [Gub10, Thm. 8.5, Prop. 6.1]. □

Remark 2.3.4. The expression $\|\mathbf{X}-\bar{\mathbf{X}}\|_\alpha$ is a slight abuse of notation since $\mathbf{X}-\bar{\mathbf{X}}$ is not a rough path in general. The definition (2.13) does however make perfect sense for the difference.

In fact, the article [Gub10] gives more precise bounds on the rough integrals than those provided in Proposition 2.3.3, but we prefer to have them in this form for the sake of conciseness.

2.4 Definition and well-posedness of the solution

Let us now give a short discussion of what we mean by "solutions" to (2.1), as introduced in [Hai11]. The idea is to find a process X such that $v = u - X$ is of class $\mathcal{C}^1$ in space, so that the definition of the integral (2.4) boils down to defining the integral

$$\int_{-\pi}^{\pi} \varphi(x)G(u(t,x))\, d_xX(t,x)\;.$$

If we have a canonical way of lifting X to a rough path $\mathbf{X}$, this integral can be interpreted in the sense of rough paths.

A natural choice for X is the solution to the linear stochastic heat equation (2.2). In order to get nice properties for this process, we build it in a slightly different way from [Hai11]. First, we define the stationary solution to the modified stochastic PDE on the circle $\mathbb{T}$,

$$dY = \Delta Y dt + \Pi dW \ , \tag{2.22}$$

where Π denotes the orthogonal projection in L^2 onto the space of functions with zero mean, i.e. it removes the zeroth Fourier mode from a Fourier expansion. In particular, if we extend the cylindrical Brownian motion W to whole $\mathbb{R}$ in time, then

$$Y(t) = \int_{-\infty}^{t} S_{t-s} \, \Pi dW(s) \ ,$$

where S is the heat semigroup defined before (2.24) below. Second, we define for all $(t, x) \in \mathbb{R}_+ \times \mathbb{T}$ the process

$$X(t,x) \stackrel{\text{def}}{=} Y(t,x) + \frac{1}{\sqrt{2\pi}} w^0(t) \ , \tag{2.23}$$

where w^0 if the zeroth Fourier mode of W, i.e. w^0 is a Brownian motion.

Remark 2.4.1. We need to use Π in (2.22) in order to obtain a stationary solution. In [Hai11], the author used instead the stationary solution to

$$dX = (\Delta X - X)dt + dW$$

as a reference path. Our choice of X was used in [HMW14] and does not change the results of [Hai11].

It follows from Lemma 2.5.1 below that there is a natural way to extend X, defined in (2.23), to a process $\mathbf{X} : \mathbb{R} \times \mathbb{T}^2 \to T^{(2)}(\mathbb{R}^n)$, such that for every fixed $t \in \mathbb{R}$, the process $\mathbf{X}(t)$ is an α-regular geometric rough path, for every $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Let us furthermore denote by $S_t \stackrel{\text{def}}{=} e^{t\Delta}$ the heat semigroup, which is given by convolution on the circle $\mathbb{T}$ with the heat kernel

$$p_t(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \sum_{k \in \mathbb{Z}} e^{-tk^2} e^{ikx} \ . \tag{2.24}$$

Assuming that the rough path-valued process $\mathbf{X}$ is given, we then *define* solutions to (2.1) as follows:

Definition 2.4.2. Setting $U(t) \stackrel{\text{def}}{=} S_t(u^0 - X(0))$, we say that u is a *mild solution* to the equation (2.1) if the function $v \stackrel{\text{def}}{=} u - X - U$ belongs to $\mathcal{C}^1_T$ for some $T > 0$ and the following identity holds

$$\begin{aligned} v(t,x) = \int_0^t S_{t-s}\big(F(u(s)) + G(u(s))\,\partial_x(v(s) + U(s))\big)(x)\,ds \\ + \int_0^t \big(S_{t-s}\partial_x Z(s)\big)(x)\,ds\;, \end{aligned} \tag{2.25}$$

for all $(t,x) \in [0,T] \times \mathbb{T}$. Here, we write for brevity $u = v + X + U$, and the function $Z(s,x)$ is a rough integral defined by

$$Z(s,x) \stackrel{\text{def}}{=} \fint_{-\pi}^{x} G(u(s,y))\,d_y X(s,y)\;, \tag{2.26}$$

whose derivative we consider in the sense of distributions.

Remark 2.4.3. In [Hai11], the last integral in (2.25) was defined by

$$\int_0^t \fint_{-\pi}^{\pi} p_{t-s}(x-y) G(u(s,y))\,d_y X(s,y)\,ds\;,$$

but as noticed in [HMW14], the notion of solution in Definition 2.4.2 is more convenient, as it simplifies treatment of the rough integral. This change does not affect the existence and uniqueness results of [Hai11], and the resulting solutions are the same.

The next theorem provides the well-posedness result for the mild solution to the equation (2.1).

Theorem 2.4.4. *Let us assume that $u^0 \in \mathcal{C}^\beta$ for some $\beta \in \left(\frac{1}{3}, \frac{1}{2}\right)$. Furthermore, let $F \in \mathcal{C}^1$ and $G \in \mathcal{C}^3$. Then for almost every realisation of the driving noise, there is $T > 0$ such that there exists a unique mild solution to* (2.1) *on the interval $[0,T]$ taking values in $\mathcal{C}\big([0,T], \mathcal{C}^\beta\big)$. If moreover, F, G and all their derivatives are bounded, then the solution is global (i.e. $T = \infty$).*

Proof. The proof can be done by performing a classical Picard iteration for v, given by (2.27), on the space $\mathcal{C}^1_T$ for some $T \leq 1$, see [Hai11]. $\square$

Remark 2.4.5. The argument of [Hai11, Thm. 3.7] also works in the space $\mathcal{C}^{1+\alpha}_{\alpha/2,T}$, for any $\alpha \in \left[0, \frac{1}{2}\right)$. Hence, the actual regularity of $v(t)$ is $1+\alpha$ rather than 1. This fact will be used in Section 2.7.1 to estimate how close the approximate derivative of v is to $\partial_x v$.

For our convenience we rewrite the mild formulation of (2.9) as

$$\bar{v} = \mathbf{F}^{\bar{v}} + \mathbf{G}^{\bar{v}} + \mathbf{Z}^{\bar{v}} - \mathbf{H}^{\bar{v}} \ , \tag{2.27}$$

where we have set the terms

$$\begin{aligned}
\mathbf{F}^{\bar{v}}(t) &\stackrel{\text{def}}{=} \int_0^t S_{t-s} F(\bar{u}(s))\, ds \ , \\
\mathbf{G}^{\bar{v}}(t) &\stackrel{\text{def}}{=} \int_0^t S_{t-s}\big(G(\bar{u}(s))\, \partial_x(\bar{v} + U)(s)\big)\, ds \ , \\
\mathbf{H}^{\bar{v}}(t)_i &\stackrel{\text{def}}{=} \Lambda \int_0^t S_{t-s} \operatorname{div} G_i(\bar{u}(s))\, ds \ , \qquad i = 1, \ldots, n \ , \\
\mathbf{Z}^{\bar{v}}(t) &\stackrel{\text{def}}{=} \int_0^t S_{t-s} \partial_x Z(s)\, ds = \int_0^t \partial_x\big(S_{t-s} Z(s)\big)\, ds \ ,
\end{aligned} \tag{2.28}$$

and as before $\bar{u} = \bar{v} + X + U$, $U(t) = S_t(u^0 - X(0))$ and

$$Z(t,x) \stackrel{\text{def}}{=} \fint_{-\pi}^{x} G(\bar{u}(t,y))\, d_y X(t,y) \ .$$

Although the two terms $\mathbf{F}^{\bar{v}}$ and $\mathbf{H}^{\bar{v}}$ are of the same type, we give them different names since they will arise in completely different ways from the approximation.

2.5 Reformulation of the solutions to the approximate equations

In this section we rewrite the mild solution to the approximate equation (2.7) in a way convenient for working in Hölder spaces of low regularity. In particular, we define the iterated integrals of higher order of the controlling process, and we include some additional terms into the equation in order to approximate the rough integral (2.26).

Similarly to (2.22) and (2.23) we define the stationary process Y_ε and X_ε by

$$dY_\varepsilon = \Delta_\varepsilon Y_\varepsilon dt + \Pi H_\varepsilon dW \;, \qquad X_\varepsilon(t,x) \stackrel{\text{def}}{=} Y_\varepsilon(t,x) + \frac{1}{\sqrt{2\pi}} w^0(t) \;, \quad (2.29)$$

where w^0 is the zeroth Fourier mode of W. Here, we have exploited Assumption 2.2.3 in order to use the same zeroth Fourier mode for X and X_ε. Moreover, we define the approximate semigroup $S_t^{(\varepsilon)} \stackrel{\text{def}}{=} e^{t\Delta_\varepsilon}$ generated by the approximate Laplacian and given by convolution on the circle $\mathbb{T}$ with the approximate heat kernel

$$p_t^{(\varepsilon)}(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}} e^{-tk^2 m(\varepsilon k)} e^{ikx} \;. \qquad (2.30)$$

Furthermore, we define $U_\varepsilon(t) \stackrel{\text{def}}{=} S_t^{(\varepsilon)}\big(u_\varepsilon^0 - X_\varepsilon(0)\big)$ and $v_\varepsilon \stackrel{\text{def}}{=} u_\varepsilon - X_\varepsilon - U_\varepsilon$. Then the mild version of the approximate equation (2.7) can be rewritten as

$$v_\varepsilon(t) = \mathbf{F}_\varepsilon^{v_\varepsilon}(t) + \mathbf{G}_\varepsilon^{v_\varepsilon}(t) + \int_0^t S_{t-s}^{(\varepsilon)}\big(G(u_\varepsilon(s))\, D_\varepsilon X_\varepsilon(s)\big)\, ds \;, \qquad (2.31)$$

where we denote for brevity $u_\varepsilon = v_\varepsilon + X_\varepsilon + U_\varepsilon$, and set

$$\begin{aligned} \mathbf{F}_\varepsilon^{v_\varepsilon}(t) &\stackrel{\text{def}}{=} \int_0^t S_{t-s}^{(\varepsilon)} F(u_\varepsilon(s))\, ds \;, \\ \mathbf{G}_\varepsilon^{v_\varepsilon}(t) &\stackrel{\text{def}}{=} \int_0^t S_{t-s}^{(\varepsilon)}\big(G(u_\varepsilon(s))\, D_\varepsilon(v_\varepsilon + U_\varepsilon)(s)\big)\, ds \;. \end{aligned} \qquad (2.32)$$

As already mentioned in Section 2.3, the rough integrals are approximated by Riemann-like sums, but these include additional higher-order correction terms. Hence, we cannot expect in general that $Z(s,x)$, defined in (2.26), is approximated by

$$\int_{-\pi}^x G(u_\varepsilon(s,y))\, D_\varepsilon X_\varepsilon(s,y)\, dy \;, \qquad (2.33)$$

as $\varepsilon \to 0$. In order to approximate $Z(s,x)$, we have to add some extra terms to (2.33). These extra terms give raise to the correction term in the limiting equation, mentioned in the introduction. In the following section we build these missing extra terms.

2.5.1 Iterated integrals of the controlling processes

In order to use the theory of rough paths with regularities close to zero, we need to build the iterated integrals of arbitrarily high orders of X and X_ε defined in (2.23) and (2.29) with respect to themselves.

The expansion of X_ε, introduced in (2.29), in the Fourier basis is given by

$$\begin{aligned} X_\varepsilon(t,x) &= \frac{1}{\sqrt{2\pi}} w_0(t) + \frac{1}{\sqrt{2\pi}} \sum_{k\in\mathbb{Z}\setminus\{0\}} \int_{-\infty}^t e^{ikx} e^{-k^2 m(\varepsilon k)(t-s)} \hbar(\varepsilon k)\, dw_k(s) \\ &= \frac{1}{\sqrt{2\pi}} w_0(t) + \frac{1}{\sqrt{\pi}} \sum_{k=1}^\infty \frac{q_k^{(\varepsilon)}}{k} \Big(\eta_k^{(\varepsilon)}(t)\sin(kx) + \eta_{-k}^{(\varepsilon)}(t)\cos(kx)\Big) \end{aligned} \tag{2.34}$$

Here, w_k are $\mathbb{C}^n$-valued standard Brownian motions (i.e. the real and imaginary parts of every component are independent real-valued Brownian motions), which are independent up to the constraint $w_k = \overline{w}_{-k}$ ensuring that X_ε is real-valued. Furthermore, for every fixed $t \geq 0$, $\eta_k^{(\varepsilon)}(t)$ are independent $\mathbb{R}^n$-valued standard Gaussian random vectors such that

$$\mathbb{E}\Big[\eta_k^{(\varepsilon)}(0) \otimes \eta_k^{(\varepsilon)}(t)\Big] = e^{-k^2 m(\varepsilon k)t}\,\mathrm{Id}\ ,$$

where Id is the $n\times n$ identity matrix, and the coefficients $q_k^{(\varepsilon)}$ are defined by

$$q_k^{(\varepsilon)} \stackrel{\text{def}}{=} \frac{\hbar(\varepsilon k)}{\sqrt{m(\varepsilon k)}}\ . \tag{2.35}$$

Similarly, the Fourier expansion of the process X defined in (2.23) is

$$X(t,x) = \frac{1}{\sqrt{2\pi}} w_0(t) + \frac{1}{\sqrt{\pi}} \sum_{k=1}^\infty \frac{1}{k} \Big(\eta_k(t)\sin(kx) + \eta_{-k}(t)\cos(kx)\Big)\ , \tag{2.36}$$

where $\eta_k(t)$ are independent $\mathbb{R}^n$-valued standard Gaussian random vectors such that

$$\mathbb{E}\big[\eta_k(0) \otimes \eta_k(t)\big] = e^{-k^2 t}\,\mathrm{Id}\ .$$

Furthermore, the random vectors $\big\{(\eta_k^{(\varepsilon)}(t), \eta_k(t)) : k \in \mathbb{Z}\setminus\{0\}\big\}$ are independent and satisfy

$$\mathbb{E}\Big[\eta_k^{(\varepsilon)}(t) \otimes \eta_k(t)\Big] = \frac{\sqrt{m(\varepsilon k)}}{m(\varepsilon k)+1}\,\mathrm{Id} \stackrel{\text{def}}{=} \tilde{q}_k^{(\varepsilon)}\,\mathrm{Id}\ .$$

The following lemma states that there are natural lifts of $X(t)$ and $X_\varepsilon(t)$ to Gaussian rough paths. The term "canonical" means here that for a large class of natural approximations of a process by smooth Gaussian processes, the respective iterated integrals converge in L^2, see [FV10] for a precise definition.

Lemma 2.5.1. *For any $\alpha \in \big(0, \frac{1}{2}\big)$ and $p = \lfloor 1/\alpha \rfloor$, there are canonical lifts $\mathbf{X}, \mathbf{X}_\varepsilon : \mathbb{R}_+ \times \mathbb{T}^2 \to T^{(p)}(\mathbb{R}^n)$ of the processes X and X_ε respectively, which are continuous functions in the time variable such that, for every $t \geq 0$, $\mathbf{X}(t)$ and $\mathbf{X}_\varepsilon(t)$ are Gaussian rough paths of regularity α. Furthermore, for any $\lambda < \frac{1}{2} - \alpha$ and any $T > 0$ the following bounds hold*

$$\mathbb{E}\big\|X\big\|_{\mathcal{C}_T^\alpha} \lesssim 1\,, \qquad \mathbb{E}\big\|X - X_\varepsilon\big\|_{\mathcal{C}_T^\alpha} \lesssim \varepsilon^\lambda\,. \tag{2.37}$$

Moreover, for any word $w \in \mathcal{A}_p$ with $|w| \geq 2$ we have

$$\mathbb{E}\big\|\mathbf{X}^w\big\|_{\mathcal{B}_T^{|w|\alpha}} \lesssim 1\,, \qquad \mathbb{E}\big\|\mathbf{X}^w - \mathbf{X}_\varepsilon^w\big\|_{\mathcal{B}_T^{|w|\alpha}} \lesssim \varepsilon^\lambda\,, \tag{2.38}$$

where we use the notation $\mathbf{X}^w \stackrel{\text{def}}{=} \langle \mathbf{X}, e_w \rangle$.

Proof. The proof of (2.37) is provided in [HMW14, Lem. 3.3]. We only have to show that there exist the claimed lifts which satisfy the estimates (2.38). To this end, we define, for some $\kappa > 0$, the following sequences

$$\beta_k^{(\varepsilon,\kappa)} \stackrel{\text{def}}{=} \frac{\hbar(\varepsilon k)^2}{k^\kappa m(\varepsilon k)}\,, \qquad \varrho_k^{(\varepsilon,\kappa)} \stackrel{\text{def}}{=} \frac{\hbar(\varepsilon k)}{k^\kappa (m(\varepsilon k)+1)}\,,$$

where $k \geq 1$. First, for the increments of $\beta_k^{(\varepsilon,\kappa)}$ we have

$$\begin{aligned}\big|\beta_{k+1}^{(\varepsilon,\kappa)} - \beta_k^{(\varepsilon,\kappa)}\big| \leq \big|\big(q_{k+1}^{(\varepsilon)}\big)^2\big|\, &\big|(k+1)^{-\kappa} - k^{-\kappa}\big| \\ &+ k^{-\kappa}\big|\big(q_{k+1}^{(\varepsilon)}\big)^2 - \big(q_k^{(\varepsilon)}\big)^2\big| \leq C k^{-1-\kappa}\,,\end{aligned}$$

for some constant $C > 0$, where $q_k^{(\varepsilon)}$ is defined in (2.35). To get the last inequality we have used the bounds on the functions m and $\hbar$, provided in Assumptions 2.2.1

and 2.2.3, and the estimate

$$\big|\big(q_{k+1}^{(\varepsilon)}\big)^2 - \big(q_k^{(\varepsilon)}\big)^2\big| \leq Ck^{-1} ,$$

which follows from the bound on the total variation of the function $\hbar^2/m$, provided by Assumption 2.2.3. Second, the convergence $\beta_k^{(\varepsilon,\kappa)} \log k \to 0$ holds as $k \to \infty$.

Using these properties of $\beta_k^{(\varepsilon,\kappa)}$, we obtain from [Tel73, Thm. 4] that the series $\sum_{k=1}^N \beta_k^{(\varepsilon,\kappa)} \cos kx$ converge in L^1 as $N \to \infty$, and the L^1-norm of the limit is independent of ε, which proves that for any $\kappa > 0$ the parametrized sequence $\beta_k^{(\varepsilon,\kappa)}$ is uniformly negligible in $\varepsilon \in (0,1]$ in the sense of [FGGR16, Def. 3.6].

Similarly, using the bound on the total variation of $\hbar/(m+1)$, which is stated in Assumption 2.2.3, we can obtain that for any $\kappa > 0$ the sequence $\varrho_k^{(\varepsilon,\kappa)}$ is uniformly negligible in $\varepsilon \in (0,1]$ as well.

Noticing that the coefficients of the Fourier expansions (2.34) and (2.36) satisfy

$$\left(\frac{q_k^{(\varepsilon)}}{k}\right)^2 = \frac{\beta_k^{(\varepsilon,\kappa)}}{k^{2-\kappa}} , \qquad \frac{q_k^{(\varepsilon)}\tilde{q}_k^{(\varepsilon)}}{k^2} = \frac{\varrho_k^{(\varepsilon,\kappa)}}{k^{2-\kappa}} ,$$

we can apply [FGGR16, Thm. 3.16] and obtain that the processes X and X_ε can indeed be lifted to rough path-valued processes $\mathbf{X}$ and $\mathbf{X}_\varepsilon$ respectively, such that for any $\alpha \in \big(0, \frac{1}{2}\big)$, for any $q \geq 1$ and for any word $w \in \mathcal{A}_p$ with $|w| \geq 2$ the bounds

$$\mathbb{E}\big\|\mathbf{X}^w(t)\big\|^q_{\mathcal{B}^{|w|\alpha}} \lesssim 1 , \qquad \mathbb{E}\big\|\mathbf{X}^w_\varepsilon(t)\big\|^q_{\mathcal{B}^{|w|\alpha}} \lesssim 1 \tag{2.39}$$

hold uniformly in $t \in [0,T]$. Furthermore, by [FGGR16, Thm. 3.17] we obtain that for all $\gamma < \frac{1}{2} - \alpha$, any $q \geq 1$ and any $\kappa > 0$ small enough,

$$\mathbb{E}\big\|\big(\mathbf{X}^w - \mathbf{X}^w_\varepsilon\big)(t)\big\|^q_{\mathcal{B}^{|w|\alpha}} \lesssim \left(\sup_{x\in\mathbb{T}} \mathbb{E}\big|(X - X_\varepsilon)(t,x)\big|^2\right)^{(\gamma+\kappa)q} \lesssim \varepsilon^{\gamma q} , \tag{2.40}$$

uniformly in $t \in [0,T]$. The last bound can be shown almost identically to [HMW14, (3.16d)], but taking $\theta \equiv 1$ and the time interval from $-\infty$.

Now we will investigate the temporal regularity of $\boldsymbol{X}_\varepsilon$. Our aim is to apply [FGGR16, Thm. 3.17] to the processes $\boldsymbol{X}_\varepsilon(s)$ and $\boldsymbol{X}_\varepsilon(t)$, with $s,t \in [0,T]$. To this end, let us define $\tau = |t-s|$ and the parametrized sequence $\mu_k^{(\tau,\varepsilon)} \stackrel{\text{def}}{=} e^{-k^2 m(\varepsilon k)\tau}$. Then, in the same way as in the beginning of the proof and using Assumptions 2.2.1

and 2.2.3, we obtain that for any $\kappa > 0$ the sequence $\beta_k^{(\kappa,\varepsilon)}\mu_k^{(\tau,\varepsilon)}$ is uniformly negligible in $\tau \in (0,1]$ and $\varepsilon \in (0,1]$ and by [FGGR16, Thm. 3.17] we obtain

$$\mathbb{E}\big\|\mathbf{X}_\varepsilon^w(t) - \mathbf{X}_\varepsilon^w(s)\big\|_{\mathcal{B}^{|w|\alpha}}^q \lesssim \left(\sup_{x\in\mathbb{T}} \mathbb{E}\big|X_\varepsilon(s,x) - X_\varepsilon(t,x)\big|^2\right)^{\gamma q} \lesssim |t-s|^{\frac{\gamma q}{2}} \ , \quad (2.41)$$

for any $\gamma < \frac{1}{2} - \alpha$, any $q \geq 1$ and any word $w \in \mathcal{A}_p$ with $|w| \geq 2$. Here, the last bound can be derived similarly to [HMW14, (3.16a)], but with $\theta \equiv 1$ and the time interval from $-\infty$. In the same way, we get

$$\mathbb{E}\big\|\mathbf{X}^w(t) - \mathbf{X}^w(s)\big\|_{\mathcal{B}^{|w|\alpha}}^q \lesssim |t-s|^{\frac{\gamma q}{2}} \ . \quad (2.42)$$

Applying the Kolmogorov criterion [Kal02] together with the bounds (2.39) and (2.42), we get the first estimate in (2.38).

Now, let us take any word $w \in \mathcal{A}_p$ with $|w| \geq 2$. Then, on the one hand, the estimate (2.40) gives for every $q \geq 1$,

$$\begin{aligned}\mathbb{E}\big\|\big(\boldsymbol{X}^w - \boldsymbol{X}_\varepsilon^w\big)(t) - \big(\boldsymbol{X}^w - \boldsymbol{X}_\varepsilon^w\big)(s)\big\|_{\mathcal{B}^{\alpha|w|}}^q \\ \lesssim \mathbb{E}\big\|\big(\boldsymbol{X}^w - \boldsymbol{X}_\varepsilon^w\big)(t)\big\|_{\mathcal{B}^{\alpha|w|}}^q + \mathbb{E}\big\|\big(\boldsymbol{X}^w - \boldsymbol{X}_\varepsilon^w\big)(s)\big\|_{\mathcal{B}^{\alpha|w|}}^q \lesssim \varepsilon^{\gamma q} \ .\end{aligned}$$

On the other hand, from (2.41) and (2.42) the next estimate follows:

$$\begin{aligned}\mathbb{E}\big\|\big(\boldsymbol{X}^w - \boldsymbol{X}_\varepsilon^w\big)(t) - \big(\boldsymbol{X}^w - \boldsymbol{X}_\varepsilon^w\big)(s)\big\|_{\mathcal{B}^{\alpha|w|}}^q \\ \lesssim \mathbb{E}\big\|\boldsymbol{X}_\varepsilon^w(t) - \boldsymbol{X}_\varepsilon^w(s)\big\|_{\mathcal{B}^{\alpha|w|}}^q + \mathbb{E}\big\|\boldsymbol{X}^w(t) - \boldsymbol{X}^w(s)\big\|_{\mathcal{B}^{\alpha|w|}}^q \lesssim |t-s|^{\frac{\gamma q}{2}} \ .\end{aligned}$$

Combining these two bunds together we obtain

$$\begin{aligned}\mathbb{E}\big\|\big(\boldsymbol{X}^w - \boldsymbol{X}_\varepsilon^w\big)(t) - \big(\boldsymbol{X}^w - \boldsymbol{X}_\varepsilon^w\big)(s)\big\|_{\mathcal{B}^{\beta|w|}}^q &\lesssim \left(\varepsilon^\gamma \wedge |t-s|^{\frac{\gamma}{2}}\right)^q \\ &\lesssim \left(\varepsilon^{\frac{1}{2}-\alpha-\delta}|t-s|^{\frac{\delta}{2}}\right)^q ,\end{aligned}$$

for any $\delta > 0$ small enough and uniformly in $s, t \in [0,T]$. From this bound, estimate (2.40) and the Kolmogorov continuity criterion [Kal02] we obtain the second bound in (2.38). $\square$

2.5.2 Approximations of the rough integral

Now, having defined the iterated integrals of X_ε and X, we can build an approximation of the process Z defined in (2.26).

The idea comes from the fact that if $u(t)$ is controlled by $\mathbf{X}(t)$, then the process $G(u(t))$ is controlled by $\mathbf{X}(t)$ as well. The Taylor expansion gives an approximation for $G_{ij}(u(t))$, for each $i, j \in \{1, \dots, n\}$, in the following way:

$$G_{ij}(u(t,y)) \approx G_{ij}(u(t,x)) + \sum_{w \in \mathcal{A}_{p-1} \setminus \emptyset} \tilde{C}_w D^w G_{ij}(u(t,x)) \big(u(t,y) - u(t,x)\big)_w ,$$

where $\tilde{C}_w$ are combinatorial factors which can be calculated explicitly. Furthermore, we have used the following notation: for $w = w_1 \cdots w_k \in \mathcal{A}_{p-1}$ and $k \geq 1$ we have denoted $D^w \stackrel{\text{def}}{=} D^{w_1} \cdots D^{w_k}$ and

$$u(t,x)_w \stackrel{\text{def}}{=} u_{w_1}(t,x) \cdots u_{w_k}(t,x) .$$

Recalling that we are looking for solutions such that $(u - X)(t) \in \mathcal{C}^1$, we obtain an approximation of $G_{ij}(u(t))$ via $\mathbf{X}(t)$,

$$G_{ij}(u(t,y)) \approx G_{ij}(u(t,x)) + \sum_{\substack{w \in \mathcal{A}_{p-1} \setminus \{\emptyset\} \\ w = w_1 \dots w_k}} \tilde{C}_w D^w G_{ij}(u(t,x)) \prod_{l=1}^{k} \langle \mathbf{X}(t;x,y), e_{w_l} \rangle .$$

Symmetrising this expression and using Definition 2.3.1, this can be rewritten as

$$G_{ij}(u(t,y)) \approx \sum_{w \in \mathcal{A}_{p-1}} C_w D^w G_{ij}(u(t,x)) \langle \mathbf{X}(t;x,y), e_w \rangle , \tag{2.43}$$

for some slightly different constants C_w. This expansion motivates our choice of the terms in the approximation of the rough integral.

In view of Assumption 2.2.2, it is natural to define for any word $w \in \mathcal{A}_p$ the process

$$\langle D_\varepsilon \mathbf{X}_\varepsilon(t;y), e_w \rangle \stackrel{\text{def}}{=} \frac{1}{\varepsilon} \int_{\mathbb{R}} \langle \mathbf{X}_\varepsilon(t;y,y+\varepsilon z), e_w \rangle \, \mu(dz) , \tag{2.44}$$

so that $D_\varepsilon \mathbf{X}_\varepsilon : \mathbb{R} \times \mathbb{T} \to T^{(p)}(\mathbb{R}^n)$. Combining the expansion (2.43) with the definition (2.18), it appears plausible that a good approximation of Z is given by

$$Z_\varepsilon(t,x)_i \stackrel{\text{def}}{=} \sum_{w\in\mathcal{A}_{p-1}} C_w \int_{-\pi}^{x} D^w G_{ij}(u_\varepsilon(t,y))\,\langle D_\varepsilon \mathbf{X}_\varepsilon(t;y), e_w\otimes e_j\rangle\,dy\;, \tag{2.45}$$

where $i,j\in\{1,\ldots,n\}$ and we have simplified the notation by omitting the sum over j.

Now we can rewrite the mild solution (2.31) as

$$v_\varepsilon = \mathbf{F}_\varepsilon^{v_\varepsilon} + \mathbf{G}_\varepsilon^{v_\varepsilon} + \mathbf{Z}_\varepsilon^{v_\varepsilon} - \mathbf{H}_\varepsilon^{v_\varepsilon} - \bar{\mathbf{H}}_\varepsilon^{v_\varepsilon}\;, \tag{2.46}$$

where the functions $\mathbf{F}_\varepsilon^{v_\varepsilon}$ and $\mathbf{G}_\varepsilon^{v_\varepsilon}$ are defined in (2.32). The term involving the rough integral is denoted by

$$\mathbf{Z}_\varepsilon^{v_\varepsilon}(t) \stackrel{\text{def}}{=} \int_0^t S_{t-s}^{(\varepsilon)}\partial_x Z_\varepsilon(s)\,ds = \int_0^t \partial_x\big(S_{t-s}^{(\varepsilon)} Z_\varepsilon(s)\big)\,ds\;. \tag{2.47}$$

The additional terms in (2.46), which we used to approximate the rough integral, are

$$\begin{aligned}
\mathbf{H}_\varepsilon^{v_\varepsilon}(t,x)_i &\stackrel{\text{def}}{=} \sum_{k\in\mathcal{A}} \int_0^t S_{t-s}^{(\varepsilon)}\Big(D^k G_{ij}(u_\varepsilon(s,\cdot))\,\langle D_\varepsilon \mathbf{X}_\varepsilon(s;\cdot), e_k\otimes e_j\rangle\Big)(x)\,ds\;,\\
\bar{\mathbf{H}}_\varepsilon^{v_\varepsilon}(t,x)_i &\stackrel{\text{def}}{=} \sum_{\substack{w\in\mathcal{A}_{p-1}\\ |w|\geq 2}} C_w \int_0^t S_{t-s}^{(\varepsilon)}\Big(D^w G_{ij}(u_\varepsilon(s,\cdot))\,\langle D_\varepsilon \mathbf{X}_\varepsilon(s;\cdot), e_w\otimes e_j\rangle\Big)(x)\,ds\;,
\end{aligned} \tag{2.48}$$

where as usual $i,j\in\{1,\ldots,n\}$ and we don't write the sum over j.

In the next sections we will show that the term $\bar{\mathbf{H}}_\varepsilon^{v_\varepsilon}$ tends to 0 and the other terms in (2.46) converge to the corresponding terms in (2.27) in respective spaces.

2.5.3 A priori bounds on the processes

In what follows we use the constant $\alpha_\star \stackrel{\text{def}}{=} \frac{1}{2}-\alpha$, for some fixed small $\alpha>0$. This constant represents the real spatial regularity of the process X defined in (2.23). To obtain better bounds we will work in the spaces of regularity α, which is close to 0. The constants α and $\alpha_\star$ are used throughout the chapter as fixed values.

To make the notation shorter, we use the seminorm introduced in (2.13) to define

$$\|\mathbf{X}\|_{\alpha_\star,T} \stackrel{\text{def}}{=} \sup_{t\in[0,T]} \|\mathbf{X}(t)\|_{\alpha_\star} . \tag{2.49}$$

Furthermore, for any $K > 0$ we define the stopping time

$$\begin{aligned}\sigma_K \stackrel{\text{def}}{=} \inf\big\{t \geq 0 \,:\, \|X\|_{\mathcal{C}_t^{\alpha_\star}} \geq K, \text{ or } \|\mathbf{X}\|_{\alpha_\star,t} \geq K, \text{ or } \|\bar{v}\|_{\mathcal{C}^{1+\alpha_\star}_{\alpha_\star/2,t}} \geq K,\\ \text{or } \|\bar{v}\|_{\mathcal{C}_t^1} \geq K\big\} .\end{aligned} \tag{2.50}$$

Note that in view of Remark 2.4.5, the condition on the norm $\|\bar{v}\|_{\mathcal{C}^{1+\alpha_\star}_{\alpha_\star/2,t}}$ is reasonable. For any two indices $i, j \in \{1, \dots, n\}$ we define the process

$$\mathcal{H}^{ij}_\varepsilon(t,x) \stackrel{\text{def}}{=} \Lambda\delta_{ij} - \langle D_\varepsilon \mathbf{X}_\varepsilon(t;x), e_i \otimes e_j\rangle ,$$

where δ is the Kronecker delta. To have a priori bounds on the corresponding ε-quantities we introduce the stopping time

$$\begin{aligned}\sigma_{K,\varepsilon} \stackrel{\text{def}}{=} \inf\big\{t \geq 0 : \|X - X_\varepsilon\|_{\mathcal{C}_t^{\alpha_\star}} \geq 1, \text{ or } \|\mathbf{X} - \mathbf{X}_\varepsilon\|_{\alpha_\star,t} \geq 1, \text{ or } \|\mathcal{H}_\varepsilon\|_{\mathcal{C}_t^{-\frac{1}{2}+\alpha}} \geq 1,\\ \text{or } \|\bar{v} - v_\varepsilon\|_{\mathcal{C}_t^\alpha} \geq 1, \text{ or } \|\bar{v} - v_\varepsilon\|_{\mathcal{C}^1_{(1-\alpha)/2,t}} \geq 1, \text{ or } \|v_\varepsilon\|_{\mathcal{C}_t^1} \geq K\big\} .\end{aligned}$$

The blow-up of the norm $\|(\bar{v} - v_\varepsilon)(t)\|_{\mathcal{C}^1}$ comes from the regularization property of the heat semigroup and the fact that we work in the α-regular spaces, i.e. we use the bound

$$\|U(t)\|_{\mathcal{C}^1} \lesssim t^{\frac{\alpha-1}{2}} \Big(\|u^0\|_{\mathcal{C}^\alpha} + \|X(0)\|_{\mathcal{C}^\alpha}\Big) ,$$

see Section 2.6 for the properties of the heat semigroup. Finally, for a sufficiently small $T > 0$ we define the stopping time

$$\varrho_{K,\varepsilon} \stackrel{\text{def}}{=} \sigma_K \wedge \sigma_{K,\varepsilon} \wedge T \tag{2.51}$$

and we write for $t \geq 0$ in what follows

$$t_\varepsilon \stackrel{\text{def}}{=} t \wedge \varrho_{K,\varepsilon} . \tag{2.52}$$

Remark 2.5.2. In this chapter we always consider time intervals up to the stopping time $\varrho_{K,\varepsilon}$. Therefore, all the quantities involved in the definition of $\varrho_{K,\varepsilon}$ are bounded by $K + 1$ and all the proportionality constants can depend on K.

Before providing a proof of Theorem 2.2.7, we establish in the following sections certain bounds on the terms of (2.27) and (2.46).

2.6 Regularity properties of the semigroups

In this section we list some properties of the heat semigroup $S_t \stackrel{\text{def}}{=} e^{t\Delta}$, defined as a convolution on the circle $\mathbb{T}$ with the heat kernel (2.24), and the approximate heat semigroup $S^{(\varepsilon)}_t \stackrel{\text{def}}{=} e^{t\Delta_\varepsilon}$, which is defined as a convolution on the circle with the approximate heat kernel (2.30).

The following lemma provides the regularising property of the heat semigroup S_t in the Hölder/Besov spaces.

Lemma 2.6.1. *If $\alpha < \beta$ and $\beta \geq 0$, then for every $t > 0$ one has the bound*

$$\|S_t\|_{\mathcal{C}^\alpha \to \mathcal{C}^\beta} \lesssim t^{\frac{\alpha-\beta}{2}} .$$

For $\alpha \leq 0$ and integer β, one can easily show this bound by the definition of the Hölder spaces. For non-integer β the bound follows by interpolation. A proof of this lemma for $\alpha \geq 0$ and $\beta \leq \alpha + 1$ can be found in [GIP15, Lem. 47]. For larger values of β, the estimate can be shown by using the semigroup property of S_t.

The following results provide the regularizing properties of the approximate semigroup $S^{(\varepsilon)}$. All the missing proofs can be found in [HMW14, Sec. 6]. We assume that Assumption 2.2.1 holds in order to derive these bounds. First, we give a bound on the difference between S_t and $S^{(\varepsilon)}_t$.

Lemma 2.6.2. *Let $\lambda \in [0, 1]$ and $\alpha \leq \gamma + \lambda$. Then for every $\kappa > 0$ sufficiently small and every $t > 0$ one has the bound*

$$\big\|S_t - S^{(\varepsilon)}_t\big\|_{\mathcal{C}^\alpha \to \mathcal{C}^\gamma} \lesssim t^{-\frac{1}{2}(\gamma-\alpha+\lambda+\kappa)} \varepsilon^\lambda .$$

Remark 2.6.3. In order to prove the bounds on the approximate semigroup, the authors in [HMW14] derived the respective bounds in the L^p-based Sobolev spaces and then exploited the Sobolev embeddings, which led to the loss of an arbitrarily small exponent κ. In case of spatial discretisations, these bounds can be improved to $\kappa = 0$ by using direct estimates on the discrete and continuous heat kernels, similarly to how it is done in Section 4.2.2.

The bounds on $S^{(\varepsilon)}$ provided by following result are analogous to the regularisation property of the heat semigroup.

Lemma 2.6.4. *For any values $\gamma, \bar{\gamma} \geq 0$ and any $t > 0$ one has the bound*

$$\sup_{\varepsilon \in (0,1]} \big\|S_t^{(\varepsilon)}\big\|_{\mathcal{C}^{\bar{\gamma}} \to \mathcal{C}^{\bar{\gamma}+\gamma-\kappa}} \lesssim t^{-\frac{\gamma}{2}} ,$$

where $\kappa > 0$ can be taken arbitrarily small.

2.7 Bounds on the approximate solutions

In this section we estimate the terms in the mild formulations of the equations (2.27) and (2.46).

2.7.1 Estimates on the reaction terms

In this section we prove convergence of the reaction terms of the approximate equation (2.46) to the corresponding terms of (2.27). Let us recall the notation (2.52) and Remark 2.5.2, which says that all the quantities involved in the definition of the stopping time $\varrho_{K,\varepsilon}$ in (2.51) are bounded on the interval $(0, t_\varepsilon]$ by the constant $K + 1$ and all the proportionality constants in the estimates below can depend on K. Furthermore, we will usually use the bounds from Section 2.6 without making references to them.

The next proposition gives a bound on the terms $\mathbf{G}^{\bar{v}}$ and $\mathbf{G}_\varepsilon^{v_\varepsilon}$ defined in (2.28) and (2.32) respectively.

Proposition 2.7.1. *For any $\gamma \in (0, 1]$, any $t > 0$ and any $\kappa > 0$ small enough the following bound holds:*

$$\begin{aligned}\big\|\big(\mathbf{G}^{\bar{v}} - \mathbf{G}_\varepsilon^{v_\varepsilon}\big)(t_\varepsilon)\|_{\mathcal{C}^\gamma} \lesssim t_\varepsilon^{\frac{1+\alpha-\gamma}{2}} \Big(&\|\bar{v} - v_\varepsilon\|_{\mathcal{C}^\alpha_{t_\varepsilon}} + \|\bar{v} - v_\varepsilon\|_{\mathcal{C}^1_{(1-\alpha)/2, t_\varepsilon}}\Big) \\ &+ \|X - X_\varepsilon\|_{\mathcal{C}^\alpha_{t_\varepsilon}} + \|u^0 - u^0_\varepsilon\|_{\mathcal{C}^\alpha} + \varepsilon^{\alpha_\star - \kappa} ,\end{aligned} \tag{2.53}$$

where we have used the constant $\alpha_\star$ defined in the beginning of Section 2.5.3.

Proof. For any $t > 0$, using the notation (2.52), we can rewrite

$$\big(\mathbf{G}^{\bar{v}} - \mathbf{G}_\varepsilon^{v_\varepsilon}\big)(t_\varepsilon) = \int_0^{t_\varepsilon} S_{t_\varepsilon - s}\big(G(\bar{u}(s))\,(\partial_x \bar{v} - D_\varepsilon \bar{v})(s)\big)\,ds$$

$$
\begin{aligned}
&+ \int_0^{t_\varepsilon} S_{t_\varepsilon - s}\big(G(\bar u(s))\,(\partial_x U - D_\varepsilon U)(s)\big)\,ds \\
&+ \int_0^{t_\varepsilon} S_{t_\varepsilon - s}\big(G(\bar u(s))\,D_\varepsilon(\bar v - v_\varepsilon)(s)\big)\,ds \\
&+ \int_0^{t_\varepsilon} S_{t_\varepsilon - s}\big(G(\bar u(s))\,D_\varepsilon(U - U_\varepsilon)(s)\big)\,ds \\
&+ \int_0^{t_\varepsilon} S_{t_\varepsilon - s}\big((G(\bar u) - G(u_\varepsilon))(s)\,D_\varepsilon(v_\varepsilon + U_\varepsilon)(s)\big)\,ds \\
&+ \int_0^{t_\varepsilon} \big(S_{t_\varepsilon - s} - S^{(\varepsilon)}_{t_\varepsilon - s}\big)\big(G(u_\varepsilon(s))\,D_\varepsilon(v_\varepsilon + U_\varepsilon)(s)\big)\,ds \\
\stackrel{\text{def}}{=} &\sum_{1 \le j \le 6} J_j \;.
\end{aligned}
$$

To bound the term J_1, we first investigate how good the operator D_ε approximates the derivative ∂_x. To this end, for a function $\varphi \in \mathcal{C}^{1+\alpha_\star}$ we use Assumption 2.2.2 and write

$$
\big(D_\varepsilon - \partial_x\big)\varphi(x) = \frac{1}{\varepsilon} \int_{\mathbb{R}} \Big(\varphi(x + \varepsilon y) - \varphi(x) - \partial_x \varphi(x) \varepsilon y\Big)\,\mu(dy) \;.
$$

From the fact that the Hölder regularity of φ is $1 + \alpha_\star$ we obtain

$$
\big|\varphi(x + \varepsilon y) - \varphi(x) - \partial_x \varphi(x) \varepsilon y\big| \lesssim |\varepsilon y|^{1+\alpha_\star} \|\varphi\|_{\mathcal{C}^{1+\alpha_\star}} \;.
$$

This immediately yields the estimate

$$
\|(D_\varepsilon - \partial_x)\varphi\|_{\mathcal{C}^0} \lesssim \varepsilon^{\alpha_\star} \|\varphi\|_{\mathcal{C}^{1+\alpha_\star}} \;, \tag{2.54}
$$

where we have used the boundedness of the $(1 + \alpha_\star)$-th moment of μ. Using this estimate we derive

$$
\begin{aligned}
\|J_1\|_{\mathcal{C}^\gamma} &\le \int_0^{t_\varepsilon} \|S_{t_\varepsilon - s}\|_{\mathcal{C}^0 \to \mathcal{C}^\gamma} \|G(\bar u(s))\|_{\mathcal{C}^0} \|(\partial_x - D_\varepsilon)\bar v(s)\|_{\mathcal{C}^0}\,ds \\
&\lesssim \varepsilon^{\alpha_\star} \int_0^{t_\varepsilon} (t_\varepsilon - s)^{-\frac{\gamma}{2}} \|\bar v(s)\|_{\mathcal{C}^{1+\alpha_\star}}\,ds \lesssim \varepsilon^{\alpha_\star} t_\varepsilon^{1 - \frac{\gamma + \alpha_\star}{2}} \;,
\end{aligned}
$$

where we have used boundedness of $\|\bar u\|_{\mathcal{C}^0_{t_\varepsilon}}$ and $\|\bar v\|_{\mathcal{C}^{1+\alpha_\star}_{\alpha_\star/2, t_\varepsilon}}$.

In order to derive a bound on J_2, we notice that

$$\|U(s)\|_{\mathcal{C}^{1+\alpha_\star}} \lesssim s^{-\frac{1}{2}}\Big(\|u^0\|_{\mathcal{C}^{\alpha_\star}} + \|X(0)\|_{\mathcal{C}^{\alpha_\star}}\Big)\;,$$

which follows from Lemma 2.6.1. Hence, using the estimate (2.54) for U, we obtain

$$\begin{aligned}\|J_2\|_{\mathcal{C}^\gamma} &\leq \int_0^{t_\varepsilon} \|S_{t_\varepsilon - s}\|_{\mathcal{C}^0\to\mathcal{C}^\gamma}\|G(\bar u(s))\|_{\mathcal{C}^0}\|(\partial_x - D_\varepsilon)U(s)\|_{\mathcal{C}^0}\,ds\\ &\lesssim \varepsilon^{\alpha_\star}\int_0^{t_\varepsilon}(t_\varepsilon - s)^{-\frac{\gamma}{2}}\|U(s)\|_{\mathcal{C}^{1+\alpha_\star}}\,ds \lesssim \varepsilon^{\alpha_\star} t_\varepsilon^{\frac{1-\gamma}{2}}\;.\end{aligned}$$

We note that for any function $\varphi \in \mathcal{C}^1$ we have, by Assumption 2.2.2,

$$|D_\varepsilon\varphi(x)| \leq \frac{1}{\varepsilon}\int_{\mathbb{R}}\int_0^{\varepsilon|z|}|\partial_x\varphi(x+y)|\,dy\,|\mu|(dz) \lesssim \|\varphi\|_{\mathcal{C}^1}\;, \tag{2.55}$$

which yields the following estimates:

$$\begin{aligned}\|J_3\|_{\mathcal{C}^\gamma} &\leq \int_0^{t_\varepsilon}\|S_{t_\varepsilon-s}\|_{\mathcal{C}^0\to\mathcal{C}^\gamma}\|G(\bar u(s))\|_{\mathcal{C}^0}\|D_\varepsilon(\bar v - v_\varepsilon)(s)\|_{\mathcal{C}^0}\,ds\\ &\lesssim \|\bar v - v_\varepsilon\|_{\mathcal{C}^1_{(1-\alpha)/2,t_\varepsilon}}\int_0^{t_\varepsilon}(t_\varepsilon - s)^{-\frac{\gamma}{2}}s^{\frac{\alpha-1}{2}}\,ds\\ &\lesssim t_\varepsilon^{\frac{1+\alpha-\gamma}{2}}\|\bar v - v_\varepsilon\|_{\mathcal{C}^1_{(1-\alpha)/2,t_\varepsilon}}\;,\end{aligned}$$

where we have used boundedness of $\|\bar u\|_{\mathcal{C}^0_{t_\varepsilon}}$.

In order to bound J_4 we use the following intermediate estimate:

$$\begin{aligned}\|(U - U_\varepsilon)(s)\|_{\mathcal{C}^1} &\leq \|S_s(u^0 - u^0_\varepsilon)\|_{\mathcal{C}^1} + \|S_s(X - X_\varepsilon)(0)\|_{\mathcal{C}^1}\\ &\qquad + \big\|\big(S_s - S_s^{(\varepsilon)}\big)\big(u^0_\varepsilon - X_\varepsilon(0)\big)\big\|_{\mathcal{C}^1}\\ &\lesssim s^{\frac{\alpha-1}{2}}\Big(\|u^0 - u^0_\varepsilon\|_{\mathcal{C}^\alpha} + \|(X - X_\varepsilon)(0)\|_{\mathcal{C}^\alpha}\Big)\\ &\qquad + s^{-\frac{1}{2}}\varepsilon^{\alpha_\star-\kappa}\Big(\|u^0_\varepsilon\|_{\mathcal{C}^{\alpha_\star}} + \|X_\varepsilon(0)\|_{\mathcal{C}^{\alpha_\star}}\Big),\end{aligned} \tag{2.56}$$

for any $\kappa > 0$ sufficiently small. Here, in the last estimate we used Lemma 2.6.2 with $\lambda = \alpha_\star - \kappa$. Using this estimate and (2.55) we obtain

$$\|J_4\|_{\mathcal{C}^\gamma} \leq \int_0^{t_\varepsilon}\|S_{t_\varepsilon-s}\|_{\mathcal{C}^0\to\mathcal{C}^\gamma}\|G(\bar u(s))\|_{\mathcal{C}^0}\|D_\varepsilon(U - U_\varepsilon)(s)\|_{\mathcal{C}^0}\,ds$$

$$\lesssim \int_0^{t_\varepsilon} (t_\varepsilon - s)^{-\frac{\gamma}{2}} \|(U - U_\varepsilon)(s)\|_{\mathcal{C}^1}\, ds$$
$$\lesssim t_\varepsilon^{\frac{1+\alpha-\gamma}{2}} \Big(\|u^0 - u^0_\varepsilon\|_{\mathcal{C}^\alpha} + \|(X - X_\varepsilon)(0)\|_{\mathcal{C}^\alpha} \Big) + \varepsilon^{\alpha_\star - \kappa} \,.$$

Exploiting continuous differentiability of the function G we obtain the following bound on the term J_5:

$$\begin{aligned}
\|J_5\|_{\mathcal{C}^\gamma} &\leq \int_0^{t_\varepsilon} \|S_{t_\varepsilon - s}\|_{\mathcal{C}^0 \to \mathcal{C}^\gamma} \|(G(\bar{u}) - G(u_\varepsilon))(s)\|_{\mathcal{C}^0} \|D_\varepsilon(v_\varepsilon + U_\varepsilon)(s)\|_{\mathcal{C}^0}\, ds \\
&\lesssim \int_0^{t_\varepsilon} (t_\varepsilon - s)^{-\frac{\gamma}{2}} s^{\frac{\alpha_\star - 1 - \kappa}{2}} \|(\bar{u} - u_\varepsilon)(s)\|_{\mathcal{C}^0}\, ds \\
&\lesssim t_\varepsilon^{\frac{1+\alpha_\star - \gamma - \kappa}{2}} \|\bar{v} - v_\varepsilon\|_{\mathcal{C}^\alpha_{t_\varepsilon}} + \|X - X_\varepsilon\|_{\mathcal{C}^\alpha_{t_\varepsilon}} + \|u^0 - u^0_\varepsilon\|_{\mathcal{C}^\alpha} + \varepsilon^{\alpha_\star - \kappa} \,,
\end{aligned}$$

where in the second line we have used a bound similar to (2.56),

$$\|D_\varepsilon U_\varepsilon(s)\|_{\mathcal{C}^0} \lesssim \|U_\varepsilon(s)\|_{\mathcal{C}^1} \lesssim s^{\frac{\alpha_\star - 1 - \kappa}{2}} \,. \tag{2.57}$$

Moreover, in the estimate on J_5 we have used the bound

$$\|(U - U_\varepsilon)(s)\|_{\mathcal{C}^0} \lesssim \|u^0 - u^0_\varepsilon\|_{\mathcal{C}^0} + \|(X - X_\varepsilon)(0)\|_{\mathcal{C}^0} + \varepsilon^{\alpha_\star - \kappa} \,, \tag{2.58}$$

which is obtained in a way similar to (2.56).

Using Lemma 2.6.2, the integral J_6 can be bounded by

$$\begin{aligned}
\|J_6\|_{\mathcal{C}^\gamma} &\leq \int_0^{t_\varepsilon} \big\|S_{t_\varepsilon - s} - S^{(\varepsilon)}_{t_\varepsilon - s}\big\|_{\mathcal{C}^0 \to \mathcal{C}^\gamma} \|G(u_\varepsilon(s))\|_{\mathcal{C}^0} \|D_\varepsilon(v_\varepsilon + U_\varepsilon)(s)\|_{\mathcal{C}^0}\, ds \\
&\lesssim \varepsilon^{\alpha_\star - \kappa} \int_0^{t_\varepsilon} (t_\varepsilon - s)^{-\frac{\alpha_\star + \gamma - \kappa/2}{2}} s^{\frac{\alpha_\star - 1 - \kappa/2}{2}}\, ds \lesssim t_\varepsilon^{\frac{1-\gamma}{2}} \varepsilon^{\alpha_\star - \kappa} \,,
\end{aligned}$$

where we have used the estimate (2.57).

Combining all these bounds together, we obtain the required result (2.53). □

In the following proposition we provide a bound on the terms $\mathbf{F}^{\bar{v}}$ and $\mathbf{F}^{v_\varepsilon}_\varepsilon$ defined in (2.28) and (2.32) respectively.

Proposition 2.7.2. *For any $\gamma \in (0, 1]$, any $t > 0$ and any $\kappa > 0$ small enough the* *following bound holds:*

$$\|(\mathbf{F}^{\bar{v}} - \mathbf{F}_\varepsilon^{v_\varepsilon})(t_\varepsilon)\|_{\mathcal{C}^\gamma} \lesssim t_\varepsilon^{1-\frac{\gamma}{2}} \|\bar{v} - v_\varepsilon\|_{\mathcal{C}^0_{t_\varepsilon}} + \|X - X_\varepsilon\|_{\mathcal{C}^0_{t_\varepsilon}} + \|u^0 - u^0_\varepsilon\|_{\mathcal{C}^0} + \varepsilon^{\alpha_\star - \kappa} ,$$

where $\alpha_\star$ is as in the beginning of Section 2.5.3.

Proof. Using continuous differentiability of the function F, Lemma 2.6.2 and recalling that $\bar{u} = \bar{v} + X + U$, we get

$$\begin{aligned}
\|(\mathbf{F}^{\bar{v}} - \mathbf{F}_\varepsilon^{v_\varepsilon})(t_\varepsilon)\|_{\mathcal{C}^\gamma} &\leq \int_0^{t_\varepsilon} \|S_{t_\varepsilon - s}\|_{\mathcal{C}^0 \to \mathcal{C}^\gamma} \|(F(\bar{u}) - F(u_\varepsilon))(s)\|_{\mathcal{C}^0}\, ds \\
&\quad + \int_0^{t_\varepsilon} \left\|S_{t_\varepsilon - s} - S^{(\varepsilon)}_{t_\varepsilon - s}\right\|_{\mathcal{C}^0 \to \mathcal{C}^\gamma} \|F(u_\varepsilon(s))\|_{\mathcal{C}^0}\, ds \\
&\lesssim \int_0^{t_\varepsilon} (t_\varepsilon - s)^{-\frac{\gamma}{2}} \|(\bar{u} - u_\varepsilon)(s)\|_{\mathcal{C}^0}\, ds + \varepsilon^{\frac{1}{2} - \kappa} \int_0^{t_\varepsilon} (t_\varepsilon - s)^{-\frac{1}{4} - \frac{\gamma}{2}}\, ds \\
&\lesssim t_\varepsilon^{1-\frac{\gamma}{2}} \|\bar{v} - v_\varepsilon\|_{\mathcal{C}^0_{t_\varepsilon}} + \|X - X_\varepsilon\|_{\mathcal{C}^0_{t_\varepsilon}} + \|u^0 - u^0_\varepsilon\|_{\mathcal{C}^0} + \varepsilon^{\alpha_\star - \kappa} .
\end{aligned}$$

Here, we have used boundedness of $\|u_\varepsilon\|_{\mathcal{C}^0_{t_\varepsilon}}$ and the estimate (2.58). $\square$

The following lemma shows how the processes (2.44) behave in the supremum norm. In particular, it shows that they converge to 0 as soon as $|w| > 2$.

Lemma 2.7.3. *For any word $w \in \mathcal{A}_p$ and any $t \geq 0$ the following bound holds:*

$$\left\|\langle D_\varepsilon \mathbf{X}_\varepsilon(t_\varepsilon; \cdot), e_w \rangle\right\|_{\mathcal{C}^0} \lesssim \varepsilon^{|w|\alpha_\star - 1} ,$$

where we have used $\alpha_\star$ defined in the beginning of Section 2.5.3.

Proof. Since $\mathbf{X}_\varepsilon(t_\varepsilon)$ is a rough path of regularity $\alpha_\star$, we can use the third property in Definition 2.3.1 to get

$$\begin{aligned}
\left|\langle D_\varepsilon \mathbf{X}_\varepsilon(t_\varepsilon; x), e_w \rangle\right| &\leq \frac{1}{\varepsilon} \int_{\mathbb{R}} \left|\langle \mathbf{X}_\varepsilon(t_\varepsilon; x, x + \varepsilon z), e_w \rangle\right| |\mu|(dz) \\
&\lesssim \varepsilon^{|w|\alpha_\star - 1} \int_{\mathbb{R}} |z|^{|w|\alpha_\star} |\mu|(dz) \lesssim \varepsilon^{|w|\alpha_\star - 1} .
\end{aligned}$$

Here, we have used the assumption on the moments of $|\mu|$. $\square$

In the following proposition we obtain a bound on the term $\bar{\mathbf{H}}_\varepsilon^{v_\varepsilon}$ defined in (2.48).

Proposition 2.7.4. *For any $\gamma \in (0,1]$ and any $t \geq 0$ we have the estimate*

$$\big\|\bar{\mathbf{H}}_\varepsilon^{v_\varepsilon}\big\|_{\mathcal{C}^\gamma_{t_\varepsilon}} \lesssim \varepsilon^{3\alpha_\star - 1}\,,$$

where the constant $\alpha_\star$ is defined in the beginning of Section 2.5.3.

Proof. We use Lemma 2.6.4 to estimate the approximate heat semigroup, and Lemma 2.7.3 to obtain

$$\begin{aligned}\big\|\bar{\mathbf{H}}_\varepsilon^{v_\varepsilon}(t_\varepsilon)\big\|_{\mathcal{C}^\gamma} &\lesssim \sum_{\substack{w\in\mathcal{A}_{p-1}\\|w|\geq 2}} \int_0^{t_\varepsilon} (t_\varepsilon - s)^{-\frac{\gamma}{2}-\kappa} \big\|D^w G(u_\varepsilon(s))\big\|_{\mathcal{C}^0} \\ &\qquad\qquad\qquad\qquad \times \big\|\langle D_\varepsilon \mathbf{X}_\varepsilon(s;\cdot), e_w \otimes e_1\rangle\big\|_{\mathcal{C}^0} ds \\ &\lesssim \sum_{\substack{w\in\mathcal{A}_{p-1}\\|w|\geq 2}} t_\varepsilon^{1-\frac{\gamma}{2}-\kappa} \varepsilon^{(|w|+1)\alpha_\star - 1} \lesssim \varepsilon^{3\alpha_\star-1}\,,\end{aligned}$$

for $\kappa > 0$ small enough, which is the claimed bound. $\square$

2.7.2 Convergence of the correction terms

In this section we show that the term $\mathbf{H}_\varepsilon^{v_\varepsilon}$, defined in (2.48), converges to the correction term $\mathbf{H}^{\bar{v}}$ from (2.28). In view of Remark 2.5.2, we only consider time intervals up to the stopping time $\varrho_{K,\varepsilon}$, by using the notation (2.52). Moreover, we usually use the bounds from Section 2.6 without mentioning them.

2.7.2.1 A Kolmogorov criterion for distribution-valued processes

We start with proving a Kolmogorov criterion for distribution-valued processes. To this end, we define for $n \geq 1$ the Dirichlet kernel

$$\mathcal{D}_n(x) \stackrel{\text{def}}{=} \frac{1}{\sqrt{2\pi}} \sum_{|k|<2^n} e^{ikx} = \frac{1}{\sqrt{2\pi}} \frac{\sin\big((2^n - \frac{1}{2})x\big)}{\sin\big(\frac{1}{2}x\big)}\,,$$

and $\mathcal{D}_0 \equiv 1$. The following lemma provides a bound on the Dirichlet kernel $\mathcal{D}_n$ in L^p-spaces.

Lemma 2.7.5. *For every $1 < p \leq \infty$ there is a constant C, depending on p, such that the bound*

$$\|\mathcal{D}_n\|_{L^p(\mathbb{T})} \leq C 2^{\frac{n}{p'}}$$

holds for every $n \geq 0$, where p' is the conjugate exponent of p.

Proof. In the case $p = \infty$, the function can be bounded by its value at 0, which gives $|\mathcal{D}_n(x)| \leq 2^{n+1}$. If $1 < p < \infty$, then we can rewrite

$$\begin{aligned}
\|\mathcal{D}_n\|_{L^p(\mathbb{T})}^p &= \frac{1}{(2\pi)^{p/2}} \int_{-\pi}^{\pi} \left| \frac{\sin\left(\left(2^n - \frac{1}{2}\right)x\right)}{\sin\left(\frac{1}{2}x\right)} \right|^p dx \\
&= \frac{2^{n(p-1)}}{(2\pi)^{p/2}} \int_{-\pi 2^n}^{\pi 2^n} \left| \frac{\sin\left(\left(1 - 2^{-(n+1)}\right)x\right)}{2^n \sin\left(2^{-(n+1)}x\right)} \right|^p dx \ .
\end{aligned}$$

The latter integral is bounded by a constant depending on p, since the integrand can be estimated up to a constant multiplier by $1 \wedge |x|^{-p}$. That gives the claimed estimate. $\square$

Now, we provide a Kolmogorov criterion for distribution-valued processes.

Lemma 2.7.6. *Let ψ be a random field on $[0,T] \times \mathbb{T}$, such that for every $t \in [0,T]$, $\psi(t)$ is a distribution taking values in a fixed Wiener chaos. Furthermore, let us assume that for every $n \geq 0$ the n-th Littlewood-Paley block satisfies*

$$\begin{aligned}
\mathbb{E}\left[\left|\delta_n \psi(t,x)\right|^2\right] &\leq A 2^{-2n\alpha} \ , \\
\mathbb{E}\left[\left|\delta_n \psi(t,x) - \delta_n \psi(s,x)\right|^2\right] &\leq B 2^{-2n\alpha} |t-s|^\delta \ ,
\end{aligned}$$

for every $x \in \mathbb{T}$, every $t, s \in [0,T]$, and some constants $A, B > 0$, $\delta > 0$ and $\alpha \in \mathbb{R} \setminus \mathbb{N}$. Then, for any $\gamma < \alpha$, $\gamma \notin \mathbb{N}$, there is a constant C, depending on α and γ, such that

$$\mathbb{E}\|\psi\|_{\mathcal{C}_T^\gamma} \leq C(A+B)^{\frac{1}{2}} \ . \tag{2.59}$$

Proof. We can notice that $\delta_n \psi(t,x) = \mathcal{D}_n * \delta_n \psi(t,x)$, where the convolution is taken over the variable $x \in \mathbb{T}$. Therefore, the Hölder inequality yields

$$|\delta_n \psi(t,x)| \leq \|\mathcal{D}_n\|_{L^{p'}(\mathbb{T})} \|\delta_n \psi(t)\|_{L^p(\mathbb{T})} \ , \tag{2.60}$$

for any $p \geq 1$, where as before p' is the exponent conjugate of p. Since $\psi(t)$ belongs to a fixed Wiener chaos, the same is true for the Littlewood-Paley block $\delta_n\psi(t)$, and we can apply Nelson's lemma to it [Nel73], saying that every moment of $\delta_n\psi(t)$ is bounded up to a constant multiplier by its second moment. Therefore, we have

$$\mathbb{E}\|\delta_n\psi(t)\|^p_{L^p(\mathbb{T})} \lesssim \int_{\mathbb{T}} \left(\mathbb{E}|\delta_n\psi(t,x)|^2\right)^{\frac{p}{2}} dx \lesssim \left(A2^{-2n\alpha}\right)^{\frac{p}{2}} , \qquad (2.61)$$

where the proportionality constant depends on p. Combining the bounds (2.60), (2.61) together with Lemma 2.7.5 and Jensen's inequality, we derive

$$\begin{aligned}\mathbb{E}\|\delta_n\psi(t)\|^2_{L^\infty} &\leq \|\mathcal{D}_n\|^2_{L^{p'}(\mathbb{T})}\mathbb{E}\|\delta_n\psi(t)\|^2_{L^p(\mathbb{T})} \\ &\leq \|\mathcal{D}_n\|^2_{L^{p'}(\mathbb{T})}\Big(\mathbb{E}\|\delta_n\psi(t)\|^p_{L^p(\mathbb{T})}\Big)^{\frac{2}{p}} \lesssim A2^{2n(\frac{1}{p'}-\alpha)} .\end{aligned}$$

Since, as it was mentioned in Section 2.1.1, for $\gamma \notin \mathbb{N}$, the space $\mathcal{C}^\gamma$ coincides with the Besov space $\mathcal{B}^\gamma_{\infty,\infty}$, we obtain

$$\begin{aligned}\mathbb{E}\|\psi(t)\|^2_{\mathcal{C}^\gamma} = \mathbb{E}\Big[\sup_{n\geq 0} 2^{2n\gamma}\|\delta_n\psi(t)\|^2_{L^\infty}\Big] &\leq \sum_{n\geq 0} 2^{2n\gamma}\mathbb{E}\|\delta_n\psi(t)\|^2_{L^\infty} \\ &\lesssim A\sum_{n\geq 0} 2^{2n(\gamma+\frac{1}{p'}-\alpha)} ,\end{aligned}$$

which is finite if $\gamma < \alpha - \frac{1}{p'}$. Finally, we can notice that for any $\gamma < \alpha$, we can choose $p' \geq 1$ large enough such that $\gamma < \alpha - \frac{1}{p'}$, so that

$$\mathbb{E}\|\psi(t)\|^2_{\mathcal{C}^\gamma} \lesssim A , \qquad (2.62)$$

for every $\gamma < \alpha$ and for a proportionality constant depending on α and γ. Repeating the same argument for $\delta_n(\psi(t) - \psi(s))$, we derive

$$\mathbb{E}\|\psi(t) - \psi(s)\|^2_{\mathcal{C}^\gamma} \lesssim B|t-s|^\delta . \qquad (2.63)$$

Since $\psi(t)$ belongs to a fixed Wiener chaos, Nelson's lemma [Nel73] yields equivalence of moments for $\|\psi(t)\|_{\mathcal{C}^\gamma}$ and $\|\psi(t) - \psi(s)\|_{\mathcal{C}^\gamma}$, and we can finish the proof by applying the Banach space-valued version of the Kolmogorov continuity criterion [HMW14, Lem. B.3], which gives (2.59) from (2.62) and (2.63). □

2.7.2.2 Convergence of the second-order iterated integrals

It follows from (2.34), that the process X_ε can be written as the Fourier series

$$X_\varepsilon(t,x) = \sum_{k\in\mathbb{Z}} b_k^{(\varepsilon)} \xi_k(t)\, e^{ikx} \;, \tag{2.64}$$

where we use coefficients $b_k^{(\varepsilon)}$ given by

$$b_k^{(\varepsilon)} \stackrel{\text{def}}{=} \begin{cases} \frac{\hbar(\varepsilon k)}{|k|\sqrt{4\pi m(\varepsilon k)}} \;, & \text{for} \quad k \neq 0 \;, \\ \frac{1}{\sqrt{2\pi}} \;, & \text{for} \quad k = 0 \;. \end{cases}$$

Furthermore, the ξ_k with $k \neq 0$ are centered stationary $\mathbb{C}^n$-valued Gaussian processes, independent up to $\xi_k = \overline{\xi}_{-k}$, and for any $t \geq 0$ they satisfy

$$\mathbb{E}\big[\xi_k(0) \otimes \xi_k(t)\big] = \mathcal{M}_k^t \,\mathrm{Id} \;, \tag{2.65}$$

where the covariance is given by $\mathcal{M}_k^t \stackrel{\text{def}}{=} e^{-m(\varepsilon k)k^2 t}$. The process ξ_0 is a Brownian motion, independent of all the others ξ_k with $k \neq 0$. In fact, the processes ξ_k depend on ε, but to simplify the notation we will not indicate it.

To make the notation shorter we define $\mathbb{X}_\varepsilon(t)$ to be the projection of the rough path $\mathbf{X}_\varepsilon(t)$ to the second level of the tensor algebra and decompose it as $\mathbb{X}_\varepsilon = \mathbb{X}_\varepsilon^+ + \mathbb{X}_\varepsilon^-$, where for any $i, j \in \{1, \ldots, n\}$,

$$\langle \mathbb{X}_\varepsilon^\mp, e_i \otimes e_j \rangle \stackrel{\text{def}}{=} \frac{1}{2}\Big(\langle \mathbb{X}_\varepsilon, e_i \otimes e_j \rangle \mp \langle \mathbb{X}_\varepsilon, e_j \otimes e_i \rangle \Big) \;.$$

Respectively, we denote by $D_\varepsilon \mathbb{X}_\varepsilon$ the projection of the process $D_\varepsilon \mathbf{X}_\varepsilon$ defined in (2.44) to the second level of the tensor algebra.

As it was noticed in [HMW14, Sec. 3], the process $\mathbb{X}_\varepsilon$ can be represented, using the expansion (2.64), as

$$\begin{aligned} \mathbb{X}_\varepsilon(t; x, y) &= \sum_{k,l\in\mathbb{Z}\setminus\{0\}} \big(\xi_k(t) \otimes \xi_l(t)\big)\, b_k^{(\varepsilon)} b_l^{(\varepsilon)} \int_x^y \big(e^{ikz} - e^{ikx}\big) i l e^{ilz}\, dz \\ &= \sum_{k,l\in\mathbb{Z}\setminus\{0\}} \big(\xi_k(t) \otimes \xi_l(t)\big)\, b_k^{(\varepsilon)} b_l^{(\varepsilon)}\, I_{kl}(y - x) \;, \end{aligned} \tag{2.66}$$

where we have used the functions

$$I_{kl}(x) \stackrel{\text{def}}{=} \begin{cases} \frac{l}{k+l}\big(e^{i(k+l)x} - 1\big) - \big(e^{ilx} - 1\big)\;, & \text{for} \quad k + l \neq 0\;, \\ ilx - \big(e^{ilx} - 1\big)\;, & \text{for} \quad k + l = 0\;. \end{cases}$$

The sum in (2.66) is understood as the L^2-limit of the partial sums over the indices $k, l \in \mathbb{Z}$ such that $0 < |k|, |l| \leq N$, and as $N \to \infty$.

In the following proposition we provide a bound on the process $D_\varepsilon \mathbb{X}_\varepsilon$ in Besov spaces of distributions. Its proof is very similar to that of [HMW14, Prop. 4.1], with the difference that in the latter the authors work in Sobolev spaces.

Proposition 2.7.7. *For any $\gamma \in \big(0, \frac{1}{2}\big)$, any $T > 0$ and any $\kappa > 0$ small enough one has the bound*

$$\mathbb{E}\Big[\big\|D_\varepsilon \mathbb{X}_\varepsilon - \Lambda \,\mathrm{Id}\,\big\|_{\mathcal{C}_T^{-\gamma}}\Big] \lesssim \varepsilon^{\gamma-\kappa}\;,$$

where Id *is the $n \times n$ identity matrix.*

Proof. For any $z \in \mathbb{R}$ and $\varepsilon > 0$ we introduce the following quantity:

$$\Lambda_{z,\varepsilon} \stackrel{\text{def}}{=} \frac{1}{\varepsilon} \sum_{k \in \mathbb{Z}\setminus\{0\}} |b_k^{(\varepsilon)}|^2 \big(1 - \cos(\varepsilon k z)\big)\;, \tag{2.67}$$

and its integrated version $\Lambda_\varepsilon \stackrel{\text{def}}{=} \int_{\mathbb{R}} \Lambda_{z,\varepsilon}\, \mu(dz)$. Then it follows from Lemmas 2.7.8 and 2.7.9 below that

$$\begin{aligned} \mathbb{E}\Big[\big\|D_\varepsilon \mathbb{X}_\varepsilon - \Lambda_\varepsilon \,\mathrm{Id}\,\big\|_{\mathcal{C}_T^{-\gamma}}\Big] &\lesssim \int_{\mathbb{R}} \mathbb{E}\Big[\sup_{s \in [0,T]} \Big\|\frac{1}{\varepsilon}\mathbb{X}_\varepsilon(s; \cdot, \cdot + \varepsilon z) - \Lambda_{z,\varepsilon} \,\mathrm{Id}\,\Big\|_{\mathcal{C}^{-\gamma}}\Big]\, |\mu|(dz) \\ &\lesssim \varepsilon^{\gamma-\kappa} \int_{\mathbb{R}} |z|^{1+\gamma-\kappa}\, |\mu|(dz) \lesssim \varepsilon^{\gamma-\kappa}\;. \end{aligned}$$

Now, the claim will follow immediately if we show that $|\Lambda_{z,\varepsilon} - \Lambda_{z,0}| \lesssim \varepsilon |z|^2$, where

$$\Lambda_{z,0} \stackrel{\text{def}}{=} \frac{1}{2\pi} \int_{\mathbb{R}_+} \frac{(1 - \cos(zs))\,\mathscr{h}^2(s)}{s^2 \mathscr{m}(s)}\, ds\;.$$

To this end, as in [HM12, Prop. 4.6] we use the fact that for any function $f : \mathbb{R} \to \mathbb{R}$ of bounded variation we have the estimate

$$\Big|\sum_{k \in \mathbb{Z}\setminus\{0\}} \varepsilon f(\varepsilon k) - \int_{\mathbb{R}} f(s)\, ds\Big| \leq \varepsilon\big(|f|_{\mathrm{BV}} + |f(0)|\big)\;.$$

Using this fact together with Assumption 2.2.3, we obtain

$$\begin{aligned}|\Lambda_{z,\varepsilon} - \Lambda_{z,0}| &\lesssim \varepsilon\Big|s \mapsto \frac{(1-\cos(zs))\hbar^2(s)}{s^2 m(s)}\Big|_{\mathrm{BV}} + \varepsilon|z|^2 \\ &\lesssim \varepsilon\Big|\frac{\hbar^2}{m}\Big|_{L^\infty}\Big|s \mapsto \frac{1-\cos(zs)}{s^2}\Big|_{\mathrm{BV}} + \varepsilon\Big|\frac{\hbar^2}{m}\Big|_{\mathrm{BV}}\Big|s \mapsto \frac{1-\cos(zs)}{s^2}\Big|_{L^\infty} + \varepsilon|z|^2 \\ &\lesssim \varepsilon|z|^2\;,\end{aligned}$$

which is the required bound. □

The following lemma provides a bound on the antisymmetric part $\mathbb{X}_\varepsilon^-$.

Lemma 2.7.8. *For any $\gamma \in \big(0, \frac{1}{2}\big)$, any $T > 0$ and any $z \in \mathbb{R}$ one has*

$$\mathbb{E}\Big[\sup_{s\in[0,T]}\Big\|\frac{1}{\varepsilon}\mathbb{X}_\varepsilon^-(s;\cdot,\cdot+\varepsilon z)\Big\|_{\mathcal{C}^{-\gamma}}\Big] \lesssim \varepsilon^{\gamma-\kappa}|z|^{1+\gamma-\kappa}\;,$$

where $\kappa > 0$ can be taken arbitrarily small.

Proof. From (2.66), we have the following expansion

$$\mathbb{X}_\varepsilon^-(s;x,x+\varepsilon z) = \sum_{k,l\in\mathbb{Z}\setminus\{0\}} b_k^{(\varepsilon)} b_l^{(\varepsilon)} J_{kl}^-(\varepsilon z)\left(\xi_k(s)\otimes\xi_l(s)\right)e^{i(k+l)x}\;,$$

where, for $|k| \neq |l|$, the function J_{kl}^- is given by

$$J_{kl}^-(\varepsilon z) \stackrel{\text{def}}{=} \frac{1}{2}\big(I_{kl}(\varepsilon z) - I_{lk}(\varepsilon z)\big) = \frac{1}{2}(l-k)\bigg(\frac{e^{i(k+l)\varepsilon z}-1}{k+l} - \frac{e^{il\varepsilon z}-e^{ik\varepsilon z}}{l-k}\bigg),$$

and for $|k| = |l|$, the value of J_{kl}^- is the corresponding limit of the above expression. We denote the m-th Littlewood-Paley block of $\frac{1}{\varepsilon}\mathbb{X}_\varepsilon^-(s;\cdot,\cdot+\varepsilon z)$ by $P_{\varepsilon,z}^{(m)}(s,\cdot)$, i.e. we can write

$$P_{\varepsilon,z}^{(m)}(s,x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon}\sum_{(k,l)\in\mathbb{Z}_0^{(m)}} b_k^{(\varepsilon)} b_l^{(\varepsilon)} J_{kl}^-(\varepsilon z)\left(\xi_k(s)\otimes\xi_l(s)\right)e^{i(k+l)x}\;,$$

for $m \geq 1$, where we have used the set $\mathbb{Z}_0^{(m)}$ defined by

$$\mathbb{Z}_0^{(m)} \stackrel{\text{def}}{=} \big\{(k,l)\in(\mathbb{Z}\setminus\{0\})^2 : 2^{m-1} \leq |k+l| < 2^m\big\}\;. \tag{2.68}$$

The zeroth Littlewood-Paley block is given by

$$P_{\varepsilon,z}^{(0)}(s,x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon} \sum_{k\in\mathbb{Z}\setminus\{0\}} b_k^{(\varepsilon)} b_{-k}^{(\varepsilon)} J_{k,-k}^{-}(\varepsilon z) \big(\xi_k(s) \otimes \xi_{-k}(s)\big) \;.$$

To prove the claim we will use Lemma 2.7.6. To this end, we will show that for every $\eta \in \big[0, \frac{1}{2}\big]$, every $x \in \mathbb{T}$ and every $s, r \in [0,T]$ we have

$$\mathbb{E}\Big[\big|P_{\varepsilon,z}^{(m)}(r,x) - P_{\varepsilon,z}^{(m)}(s,x)\big|^2\Big] \lesssim 2^{2m\eta} \varepsilon^{2\eta-2\kappa} |z|^{2+2\eta-2\kappa} |r-s|^{\kappa} \;, \tag{2.69}$$

$$\mathbb{E}\Big[\big|P_{\varepsilon,z}^{(m)}(r,x)\big|^2\Big] \lesssim 2^{2m\eta} \varepsilon^{2\eta} |z|^{2+2\eta} \;, \tag{2.70}$$

for any $m \in \mathbb{N}$ and $\kappa > 0$ sufficiently small. Moreover, we use the fact that $\mathbb{X}_{\varepsilon}^{-}(r)$ belongs to the second Wiener chaos.

We start with proving (2.69). To this end, for $i \neq j \in \{1,\dots,n\}$ we define $\zeta_{kl}^{ij}(s,r) \stackrel{\text{def}}{=} \xi_k^i(r)\xi_l^j(r) - \xi_k^i(s)\xi_l^j(s)$. Then for any integer $m \geq 1$ we can rewrite

$$\begin{aligned} \mathbb{E}\Big[\big|P_{\varepsilon,z}^{(m)}(r,x) - P_{\varepsilon,z}^{(m)}(s,x)\big|^2\Big] = \sum_{\substack{(k,l),(\bar{k},\bar{l})\in\mathbb{Z}_0^{(m)} \\ k+l=\bar{k}+\bar{l}}} & b_k^{(\varepsilon)} b_l^{(\varepsilon)} b_{\bar{k}}^{(\varepsilon)} b_{\bar{l}}^{(\varepsilon)} \frac{1}{\varepsilon} J_{kl}^{-}(\varepsilon z) \frac{1}{\varepsilon} J_{-\bar{k},-\bar{l}}^{-}(\varepsilon z) \\ & \times \mathbb{E}\Big[\sum_{i\neq j} \zeta_{kl}^{ij}(s,r) \zeta_{-\bar{k},-\bar{l}}^{ji}(s,r)\Big] \;. \end{aligned}$$

Furthermore, for $i \neq j$, the following relation is a consequence of (2.65):

$$\begin{aligned} \mathbb{E}\Big[\xi_k^i(r)\xi_l^j(r)\xi_{-\bar{k}}^i(s)\xi_{-\bar{l}}^j(s)\Big] &= \mathbb{E}\big[\xi_k^i(r)\xi_{-\bar{k}}^i(s)\big]\mathbb{E}\Big[\xi_l^j(r)\xi_{-\bar{l}}^j(s)\Big] \\ &= \delta_{k\bar{k}}\delta_{l\bar{l}}\, \mathcal{M}_k^{r-s}\mathcal{M}_l^{r-s} \;, \end{aligned} \tag{2.71}$$

where δ is the Kronecker function. Therefore, for any $\kappa \in [0,1]$ and any indices satisfying $k+l = \bar{k}+\bar{l}$ we obtain

$$\begin{aligned} \mathbb{E}\Big[\zeta_{kl}^{ij}(s,r)\zeta_{-\bar{k},-\bar{l}}^{ji}(s,r)\Big] &= 2\delta_{k\bar{k}}\big(1 - \mathcal{M}_k^{r-s}\mathcal{M}_l^{r-s}\big) \\ &\lesssim \delta_{k\bar{k}}\Big(1 - e^{-\big(m(\varepsilon k)k^2 + m(\varepsilon l)l^2\big)|r-s|}\Big) \\ &\lesssim \delta_{k\bar{k}}\Big(m(\varepsilon k)^{\kappa}k^{2\kappa} + m(\varepsilon l)^{\kappa}l^{2\kappa}\Big)|r-s|^{\kappa} \;. \end{aligned}$$

Assumption 2.2.3 yields for any $k, l \in \mathbb{Z} \setminus \{0\}$,

$$b_k^{(\varepsilon)} b_l^{(\varepsilon)} \lesssim \frac{1}{\sqrt{m(\varepsilon k) m(\varepsilon l)}|kl|} \lesssim \frac{1}{\sqrt{m(\varepsilon k) m(\varepsilon l)}\big|(k+l)^2 - (k-l)^2\big|} \; .$$

Moreover, we can rewrite

$$\Big|\frac{1}{\varepsilon} J_{kl}^{-}(\varepsilon z)\Big|^2 = (k-l)^2 z^2 \Big(S\big((k+l)\varepsilon z\big) - S\big((k-l)\varepsilon z\big)\Big)^2 \; ,$$

where $S(x) \stackrel{\text{def}}{=} \sin(x/2)/x$. Hence, using Assumption 2.2.1, saying that the function m is bounded from below, we obtain for $\kappa \in [0, \frac{1}{2}]$,

$$\begin{aligned}
&\mathbb{E}\Big[\big|P_{\varepsilon,z}^{(m)}(r,x) - P_{\varepsilon,z}^{(m)}(s,x)\big|^2\Big] \\
&\quad\lesssim \sum_{(k,l)\in\mathbb{Z}_0^{(m)}} \big(b_k^{(\varepsilon)} b_l^{(\varepsilon)}\big)^2 \Big|\frac{1}{\varepsilon} J_{kl}^{-}(\varepsilon z)\Big|^2 \Big(m(\varepsilon k)^\kappa |k|^{2\kappa} + m(\varepsilon l)^\kappa |l|^{2\kappa}\Big) |r-s|^\kappa \\
&\quad\lesssim |r-s|^\kappa z^2 \sum_{(k,l)\in\mathbb{Z}_0^{(m)}} (k-l)^2 \left|\frac{S((k+l)\varepsilon z) - S((k-l)\varepsilon z)}{(k+l)^2 - (k-l)^2}\right|^2 \big(|k|^{2\kappa} + |l|^{2\kappa}\big) \\
&\quad\lesssim |r-s|^\kappa z^2 \int_{2^{m-1}\le|x+y|<2^m} (x-y)^2 \left|\frac{S((x+y)\varepsilon z) - S((x-y)\varepsilon z)}{(x+y)^2 - (x-y)^2}\right|^2 \\
&\qquad\qquad \times \big(|x|^{2\kappa} + |y|^{2\kappa}\big)\, dx\, dy \; .
\end{aligned}$$

Changing the variables in the integral by $(x+y)\varepsilon z \mapsto x$ and $(x-y)\varepsilon z \mapsto y$, we derive

$$\begin{aligned}
&\mathbb{E}\Big[\big|P_{\varepsilon,z}^{(m)}(r,x) - P_{\varepsilon,z}^{(m)}(s,x)\big|^2\Big] \\
&\quad\lesssim |r-s|^\kappa \varepsilon^{-2\kappa} |z|^{2-2\kappa} \int_{\mathbb{R}} \int_{2^{m-1}\varepsilon|z|}^{2^m\varepsilon|z|} y^2 \left|\frac{S(x) - S(y)}{x^2 - y^2}\right|^2 \big(|x|^{2\kappa} + |y|^{2\kappa}\big)\, dx\, dy \; .
\end{aligned}$$

Finally, using the bound

$$\left|\frac{S(x) - S(y)}{x^2 - y^2}\right| \lesssim 1 \wedge \frac{1}{x^2 + y^2} \; ,$$

we derive that the last integral is bonded up to a constant by

$$1 \wedge (2^m \varepsilon |z|) \le 2^{2m\eta} \varepsilon^{2\eta} |z|^{2\eta} \; ,$$

for any $\eta \in \left[0, \frac{1}{2}\right]$, what implies the estimate (2.69).

In order to prove (2.70) we use (2.71) and obtain

$$\mathbb{E}\Big[\big|P^{(m)}_{\varepsilon,z}(r,x)\big|^2\Big] = \sum_{\substack{(k,l),(\bar k,\bar l)\in\mathbb{Z}_0^{(m)}\\ k+l=\bar k+\bar l}} b_k^{(\varepsilon)} b_l^{(\varepsilon)} b_{\bar k}^{(\varepsilon)} b_{\bar l}^{(\varepsilon)}\, \frac{1}{\varepsilon} J^-_{kl}(\varepsilon z)\, \frac{1}{\varepsilon} J^-_{-\bar k,-\bar l}(\varepsilon z)$$

$$\times\, \mathbb{E}\Big[\sum_{i\neq j} \xi_k^i(r)\xi_l^j(r)\xi_{-\bar k}^i(r)\xi_{-\bar l}^j(r)\Big]$$

$$\lesssim \sum_{(k,l)\in\mathbb{Z}_0^{(m)}} \big(b_k^{(\varepsilon)} b_l^{(\varepsilon)}\big)^2 \Big|\frac{1}{\varepsilon} J^-_{kl}(\varepsilon z)\Big|^2\,,$$

The claimed bound follows by repeating the argument for (2.69) with $\kappa = 0$. The bounds on the zeroth block are derived respectively. $\square$

In the following lemma we derive a bound on the symmetric part $\mathbb{X}^+_\varepsilon$.

Lemma 2.7.9. *For any $\gamma \in \left(0, \frac{1}{2}\right)$, any $T > 0$, any $z \in \mathbb{R}$ and any $\kappa > 0$ small enough one has*

$$\mathbb{E}\Big[\sup_{s\in[0,T]} \Big\|\frac{1}{\varepsilon}\mathbb{X}^+_\varepsilon(s;\cdot,\cdot+\varepsilon z) - \Lambda_{z,\varepsilon}\,\mathrm{Id}\,\Big\|_{\mathcal{C}^{-\gamma}}\Big] \lesssim \varepsilon^{\gamma-\kappa}|z|^{1+\gamma-\kappa}\,,$$

where the quantity $\Lambda_{z,\varepsilon}$ is defined in (2.67).

Proof. From (2.66) we conclude that the Fourier expansion of $\mathbb{X}^+(s)$ is

$$\mathbb{X}^+_\varepsilon(s;x,x+\varepsilon z) = \sum_{k,l\in\mathbb{Z}_0} b_k^{(\varepsilon)} b_l^{(\varepsilon)}\, J^+_{kl}(\varepsilon z)\big(\xi_k(s)\otimes\xi_l(s)\big)\, e^{i(k+l)x}\,,$$

where the function J^+_{kl} is given by

$$J^+_{kl} \stackrel{\text{def}}{=} \frac{1}{2}\Big(I_{kl}(\varepsilon z) + I_{lk}(\varepsilon z)\Big) = \frac{1}{2}\Big(1 - e^{ik\varepsilon z}\Big)\Big(1 - e^{il\varepsilon z}\Big)\,.$$

The m-th Littlewood-Paley block of $\frac{1}{\varepsilon}\mathbb{X}^+_\varepsilon(s;\cdot,\cdot+\varepsilon z)$ we denote by $Q^{(m)}_{\varepsilon,z}(s,\cdot)$, i.e.

$$Q^{(m)}_{\varepsilon,z}(s,x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon}\sum_{(k,l)\in\mathbb{Z}_0^{(m)}} b_k^{(\varepsilon)} b_l^{(\varepsilon)}\, J^+_{kl}(\varepsilon z)\,\big(\xi_k(s)\otimes\xi_l(s)\big)\, e^{i(k+l)x}\,,$$

for integer $m \geq 1$, where $\mathbb{Z}_0^{(m)}$ is defined in (2.68). The zeroth Littlewood-Paley block is given by

$$Q^{(0)}_{\varepsilon,z}(s,x) \stackrel{\text{def}}{=} \frac{1}{\varepsilon} \sum_{k\in\mathbb{Z}\setminus\{0\}} |b^{(\varepsilon)}_k|^2 J^+_{k,-k}(\varepsilon z) \big(\xi_k(s)\otimes\xi_{-k}(s)\big)$$
$$= \frac{1}{\varepsilon} \sum_{k\in\mathbb{Z}\setminus\{0\}} |b^{(\varepsilon)}_k|^2 \big(1-\cos(z\varepsilon k)\big)\big(\xi_k(s)\otimes\xi_{-k}(s)\big)\;.$$

Since $\mathbb{X}^+_\varepsilon(s)$ belongs to the second Wiener chaos, we can use Lemma 2.7.6 after showing that for every $\eta \in \big[0,\frac{1}{2}\big]$, every $x\in\mathbb{T}$ and every $s,r\in[0,T]$ we have

$$\mathbb{E}\Big[\big|Q^{(m)}_{\varepsilon,z}(r,x) - Q^{(m)}_{\varepsilon,z}(s,x)\big|^2\Big] \lesssim 2^{2m\eta}\varepsilon^{2\eta-2\kappa}|z|^{2+2\eta-2\kappa}|r-s|^\kappa\;, \tag{2.72}$$
$$\mathbb{E}\Big[\big|Q^{(m)}_{\varepsilon,z}(r,x) - \Lambda_{z,\varepsilon}\delta_{m,0}\,\mathrm{Id}\big|^2\Big] \lesssim 2^{2m\eta}\varepsilon^{2\eta}|z|^{2+2\eta}\;, \tag{2.73}$$

for any $m\in\mathbb{N}$ and $\kappa>0$ sufficiently small, where δ is the Kronecker delta.

To this end, we define $\zeta_{kl}(s,r) \stackrel{\text{def}}{=} \xi_k(r)\otimes\xi_l(r) - \xi_k(s)\otimes\xi_l(s)$. Then for any integer $m\geq 1$ we have the expansion

$$\mathbb{E}\Big[\big|Q^{(m)}_{\varepsilon,z}(r,x) - Q^{(m)}_{\varepsilon,z}(s,x)\big|^2\Big] = \sum_{\substack{(k,l),(\bar k,\bar l)\in\mathbb{Z}^{(m)}_0\\ k+l=\bar k+\bar l}} b^{(\varepsilon)}_k b^{(\varepsilon)}_l b^{(\varepsilon)}_{\bar k} b^{(\varepsilon)}_{\bar l}\,\frac{1}{\varepsilon}J^+_{kl}(\varepsilon z)\,\frac{1}{\varepsilon}J^+_{-\bar k,-\bar l}(\varepsilon z)$$
$$\times\,\mathbb{E}\,\mathrm{tr}\,\big(\zeta_{kl}(s,r)\zeta_{-\bar k,-\bar l}(s,r)\big)\;.$$

By the same argument as in the proof of Lemma 2.7.8 we can show that

$$\mathbb{E}\,\mathrm{tr}\,\big(\zeta_{kl}(s,r)\zeta_{-\bar k,-\bar l}(s,r)\big) = 2\big(n^2\,\delta_{k\bar l}\delta_{\bar k l} + n\,\delta_{k\bar k}\delta_{l\bar l}\big)\big(1-\mathcal{M}^{r-s}_k\mathcal{M}^{r-s}_l\big)\;.$$

Furthermore, it follows from the definition of $\mathcal{M}$, that for any $\kappa\in[0,1]$,

$$\big|1-\mathcal{M}^{r-s}_k\mathcal{M}^{r-s}_l\big| \lesssim \Big(\mathfrak{m}(\varepsilon k)^\kappa|k|^{2\kappa} + \mathfrak{m}(\varepsilon l)^\kappa|l|^{2\kappa}\Big)|r-s|^\kappa\;.$$

As in the proof of Lemma 2.7.8, we have $b^{(\varepsilon)}_k \lesssim \big(\sqrt{\mathfrak{m}(\varepsilon k)}|k|\big)^{-1}$, for $k\neq 0$. Moreover, we can rewrite

$$\big|J^+_{kl}(\varepsilon z)\big|^2 = 2\big(1-\cos(k\varepsilon z)\big)\big(1-\cos(l\varepsilon z)\big)\;.$$

Combining all these bounds together we obtain, for $\kappa \in \big[0, \frac{1}{2}\big]$,

$$\begin{aligned}
&\mathbb{E}\Big[\big|Q^{(m)}_{\varepsilon,z}(r,x) - Q^{(m)}_{\varepsilon,z}(s,x)\big|^2\Big] \\
&\lesssim \sum_{(k,l)\in\mathbb{Z}_0^{(m)}} \big(b^{(\varepsilon)}_k b^{(\varepsilon)}_l\big)^2 \frac{1}{\varepsilon^2}\big|J^+_{kl}(\varepsilon z)\big|^2 \big|1 - \mathcal{M}_k^{r-s}\mathcal{M}_l^{r-s}\big| \\
&\lesssim \sum_{(k,l)\in\mathbb{Z}_0^{(m)}} \frac{1-\cos(k\varepsilon z)}{k^2\varepsilon}\frac{1-\cos(l\varepsilon z)}{l^2\varepsilon}\frac{\mathfrak{m}(\varepsilon k)^\kappa |k|^{2\kappa} + \mathfrak{m}(\varepsilon l)^\kappa |l|^{2\kappa}}{\mathfrak{m}(\varepsilon k)\mathfrak{m}(\varepsilon l)}|r-s|^\kappa \\
&\lesssim |r-s|^\kappa \sum_{(k,l)\in\mathbb{Z}_0^{(m)}} \frac{1-\cos(k\varepsilon z)}{k^2\varepsilon}\frac{1-\cos(l\varepsilon z)}{l^2\varepsilon}\big(|k|^{2\kappa} + |l|^{2\kappa}\big) \\
&\lesssim |r-s|^\kappa \int_{2^{m-1}\le|x+y|<2^m} \frac{1-\cos(x\varepsilon z)}{x^2\varepsilon}\frac{1-\cos(y\varepsilon z)}{y^2\varepsilon}\big(|x|^{2\kappa} + |y|^{2\kappa}\big)\,dx\,dy\;.
\end{aligned}$$

Applying the following change of the variables in the integral, $x\varepsilon z \mapsto x$ and $y\varepsilon z \mapsto y$, we derive

$$\begin{aligned}
\mathbb{E}\Big[\big|Q^{(m)}_{\varepsilon,z}(r,x) - Q^{(m)}_{\varepsilon,z}(s,x)\big|^2\Big] &\lesssim |r-s|^\kappa \varepsilon^{-2\kappa}|z|^{2-2\kappa} \\
&\quad\times \int_{\mathbb{R}}\int_{x+2^{m-1}\varepsilon|z|}^{x+2^m\varepsilon|z|} \frac{1-\cos(x)}{x^2}\frac{1-\cos(y)}{y^2}\big(|x|^{2\kappa} + |y|^{2\kappa}\big)\,dy\,dx\;.
\end{aligned}$$

Now we can use the simple bound

$$\frac{1-\cos(x)}{x^2} \lesssim 1 \wedge \frac{1}{x^2}\;,$$

to obtain that the integral in y is bounded by $1 \wedge (2^m\varepsilon|z|) \le 2^{2m\eta}\varepsilon^{2\eta}|z|^{2\eta}$, for any $\eta \in \big[0, \frac{1}{2}\big]$, what implies the estimate (2.72). The zeroth block is bounded in the same way.

In order to show (2.73), we first consider the case $m \ge 1$. We define for brevity $\hat{\zeta}_{kl}(r) \stackrel{\text{def}}{=} \xi_k(r) \otimes \xi_l(r)$. Then we can rewrite

$$\begin{aligned}
\mathbb{E}\Big[\big|Q^{(m)}_{\varepsilon,z}(r,x)\big|^2\Big] = \sum_{\substack{(k,l),(\bar{k},\bar{l})\in\mathbb{Z}_0^{(m)}\\ k+l=\bar{k}+\bar{l}}} & b^{(\varepsilon)}_k b^{(\varepsilon)}_l b^{(\varepsilon)}_{\bar{k}} b^{(\varepsilon)}_{\bar{l}}\, \frac{1}{\varepsilon}J^+_{kl}(\varepsilon z)\,\frac{1}{\varepsilon}J^+_{-\bar{k},-\bar{l}}(\varepsilon z) \\
&\times \mathbb{E}\,\mathrm{tr}\Big(\hat{\zeta}_{kl}(r)\hat{\zeta}_{-\bar{k},-\bar{l}}(r)\Big)\;.
\end{aligned}$$

A case by case argument yields the identity

$$\mathbb{E}\,\mathrm{tr}\left(\hat{\zeta}_{kl}(r)\hat{\zeta}_{-\bar{k},-\bar{l}}(r)\right) = n^2\,\delta_{k\bar{l}}\delta_{\bar{k}l} + n\,\delta_{k\bar{k}}\delta_{l\bar{l}}\;.$$

Furthermore, performing the estimates as above we obtain, for any $\eta \in \left[0, \frac{1}{2}\right]$,

$$\begin{aligned}
\mathbb{E}\left[\left|Q^{(m)}_{\varepsilon,z}(r,x)\right|^2\right] &\lesssim \sum_{(k,l)\in\mathbb{Z}^{(m)}_0} \left(b^{(\varepsilon)}_k b^{(\varepsilon)}_l\right)^2 \frac{1}{\varepsilon^2}\left|J^+_{kl}(\varepsilon z)\right|^2 \\
&\lesssim \sum_{(k,l)\in\mathbb{Z}^{(m)}_0} \frac{1-\cos(k\varepsilon z)}{\mathfrak{m}(\varepsilon k)k^2\varepsilon}\frac{1-\cos(l\varepsilon z)}{\mathfrak{m}(\varepsilon k)l^2\varepsilon} \\
&\lesssim \sum_{(k,l)\in\mathbb{Z}^{(m)}_0} \frac{1-\cos(k\varepsilon z)}{k^2\varepsilon}\frac{1-\cos(l\varepsilon z)}{l^2\varepsilon} \\
&\lesssim 2^{m\eta}\varepsilon^{2\eta}|z|^{2+2\eta}\;,
\end{aligned}$$

Now, we will show (2.73) with $m = 0$. To this end, we define the following function: $\tilde{\zeta}_k(r) \stackrel{\text{def}}{=} \xi_k(r)\otimes\xi_{-k}(r) - \mathrm{Id}$. Using this notation we can rewrite

$$\begin{aligned}
\mathbb{E}\left[\left|Q^{(0)}_{\varepsilon,z}(r,x) - \Lambda_{\varepsilon,z}\,\mathrm{Id}\right|^2\right] = \sum_{k,\bar{k}\in\mathbb{Z}\setminus\{0\}} |b^{(\varepsilon)}_k|^2|b^{(\varepsilon)}_{\bar{k}}|^2 \frac{1-\cos(z\varepsilon k)}{\varepsilon}\frac{1-\cos(z\varepsilon\bar{k})}{\varepsilon} \\
\times\,\mathbb{E}\,\mathrm{tr}\left(\tilde{\zeta}_k(r)\tilde{\zeta}_{-\bar{k}}(r)\right)\;.
\end{aligned}$$

Moreover, it is easy to show that $\mathbb{E}\,\mathrm{tr}\left(\tilde{\zeta}_k(r)\tilde{\zeta}_{-\bar{k}}(r)\right) \lesssim 1$. Hence, using the bounds as above we derive

$$\mathbb{E}\left[\left|Q^{(0)}_{\varepsilon,z}(r,x) - \Lambda_{\varepsilon,z}\,\mathrm{Id}\right|^2\right] \lesssim \sum_{k,\bar{k}\in\mathbb{Z}\setminus\{0\}} \frac{1-\cos(k\varepsilon z)}{k^2\varepsilon}\frac{1-\cos(\bar{k}\varepsilon z)}{\bar{k}^2\varepsilon} \lesssim \varepsilon^{2\eta}|z|^{2+2\eta}\;,$$

where $\eta \in \left[0, \frac{1}{2}\right]$, which finishes the proof. □

2.7.2.3 Bounds on the correction terms

A bound on the terms $\mathbf{H}^{\bar{v}}$ and $\mathbf{H}^{v_\varepsilon}_\varepsilon$, defined in (2.28) and (2.48) respectively, is given in the following proposition.

Proposition 2.7.10. *For any $\gamma \in (0, 1]$, any $t > 0$ and any $\kappa > 0$ sufficiently small* *one has the bound*

$$\mathbb{E}\big\|\mathbf{H}^{\bar{v}} - \mathbf{H}^{v_\varepsilon}_\varepsilon\big\|_{\mathcal{C}^\gamma_{t_\varepsilon}} \lesssim T^{1-\frac{\gamma}{2}}\mathbb{E}\|\bar{v} - v_\varepsilon\|_{\mathcal{C}^0_{t_\varepsilon}} + \mathbb{E}\|X - X_\varepsilon\|_{\mathcal{C}^0_{t_\varepsilon}} + \mathbb{E}\|u^0 - u^0_\varepsilon\|_{\mathcal{C}^0} + \varepsilon^{\alpha_\star-\kappa},$$

where $T > 0$ *is as in* (2.51) *and the constant* $\alpha_\star$ *is defined in the beginning of Section 2.5.3.*

Proof. For each $i \in \{1, \ldots, n\}$, let us define the functions $\mathcal{F}(u)_i \stackrel{\text{def}}{=} \Lambda \operatorname{div} G_i(u)$ and

$$\mathcal{F}_\varepsilon(u)_i(s, x) \stackrel{\text{def}}{=} \sum_{w\in\mathcal{A}} D^w G_{ij}(u(s, x))\langle D_\varepsilon \mathbf{X}_\varepsilon(s, x), e_w \otimes e_j\rangle \;,$$

where as usual the sum over $j \in \{1, \ldots, n\}$ is omitted. Then we can write

$$\begin{aligned}
(\mathbf{H}^{\bar{v}} - \mathbf{H}^{v_\varepsilon}_\varepsilon)(t_\varepsilon) &= \int_0^{t_\varepsilon} S_{t_\varepsilon-s}\big(\mathcal{F}(u_\varepsilon) - \mathcal{F}_\varepsilon(u_\varepsilon)\big)(s)\, ds \\
&\quad + \int_0^{t_\varepsilon} S_{t_\varepsilon-s}\big(\mathcal{F}(\bar{u}) - \mathcal{F}(u_\varepsilon)\big)(s)\, ds + \int_0^{t_\varepsilon} \big(S_{t_\varepsilon-s} - S^{(\varepsilon)}_{t_\varepsilon-s}\big)\mathcal{F}_\varepsilon(u_\varepsilon)(s)\, ds \\
&\stackrel{\text{def}}{=} J_1 + J_2 + J_3 \;.
\end{aligned}$$

In order to bound the first term J_1 we note that we can rewrite

$$\big(\mathcal{F}(u_\varepsilon) - \mathcal{F}_\varepsilon(u_\varepsilon)\big)_i(s, x) = \sum_{w\in\mathcal{A}} D^w G_{ij}(u_\varepsilon(s, x))\Big(\Lambda\, \delta_{w,j} - \langle D_\varepsilon \mathbf{X}_\varepsilon(s, x), e_w \otimes e_j\rangle\Big) \;,$$

where δ is the Kronecker function. Therefore, applying Lemma 2.6.1 with $\eta \in (0, \alpha_\star)$ and Lemma 2.1.1, we obtain

$$\begin{aligned}
\|J_1\|_{\mathcal{C}^\gamma} &\lesssim \int_0^{t_\varepsilon} \|S_{t_\varepsilon-s}\|_{\mathcal{C}^{-\eta}\to\mathcal{C}^\gamma} \big\|\big(\mathcal{F}(u_\varepsilon) - \mathcal{F}_\varepsilon(u_\varepsilon)\big)(s)\big\|_{\mathcal{C}^{-\eta}}\, ds \\
&\lesssim \sup_{s\in[0,t_\varepsilon]} \big\|D_\varepsilon \mathbb{X}_\varepsilon(s, \cdot) - \Lambda\, \mathrm{Id}\,\big\|_{\mathcal{C}^{-\eta}} \|DG(u_\varepsilon)\|_{\mathcal{C}^{\alpha_\star}_{t_\varepsilon}} \int_0^{t_\varepsilon} (t_\varepsilon - s)^{-\frac{\eta+\gamma}{2}}\, ds \;.
\end{aligned}$$

That gives us, using the boundedness of $\|u_\varepsilon\|_{\mathcal{C}^{\alpha_\star}_{t_\varepsilon}}$ and Proposition 2.7.7,

$$\mathbb{E}\|J_1\|_{\mathcal{C}^\gamma_{t_\varepsilon}} \lesssim T^{1-\frac{\eta+\gamma}{2}}\varepsilon^{\eta-\kappa} \;,$$

where $T > 0$ is as in (2.51). A bound on J_2 follows from Lemma 2.6.1 and regularity of G,

$$
\begin{aligned}
\|J_2\|_{\mathcal{C}^\gamma} &\lesssim \int_0^{t_\varepsilon} (t_\varepsilon - s)^{-\frac{\gamma}{2}} \big\| \big(\mathcal{F}(\bar{u}) - \mathcal{F}(u_\varepsilon)\big)(s) \big\|_{\mathcal{C}^0} \, ds \\
&\lesssim \int_0^{t_\varepsilon} (t_\varepsilon - s)^{-\frac{\gamma}{2}} \|(\bar{u} - u_\varepsilon)(s)\|_{\mathcal{C}^0} \, ds \\
&\lesssim t_\varepsilon^{1-\frac{\gamma}{2}} \|\bar{v} - v_\varepsilon\|_{\mathcal{C}^0_{t_\varepsilon}} + \|X - X_\varepsilon\|_{\mathcal{C}^0_{t_\varepsilon}} + \|u^0 - u^0_\varepsilon\|_{\mathcal{C}^0} + \varepsilon^{\alpha_\star - \kappa} \; .
\end{aligned}
$$

Here, we have used the representation of $\bar{u}$ via $\bar{v}$ and the bound (2.58).

For the third term J_3 we use Lemma 2.6.2 with $\lambda = \frac{1}{2} - \kappa$ and get

$$
\|J_3\|_{\mathcal{C}^\gamma_{t_\varepsilon}} \lesssim \varepsilon^{\frac{1}{2}-\kappa} t_\varepsilon^{1-\frac{1}{2}\left(\gamma+\frac{1}{2}\right)} \|\mathcal{F}_\varepsilon(u_\varepsilon)\|_{\mathcal{C}^0_{t_\varepsilon}} \lesssim \varepsilon^{\frac{1}{2}-\kappa} t_\varepsilon^{\frac{3}{4}-\frac{\gamma}{2}} \; ,
$$

where we have used boundedness of the second-order iterated integral $\mathbb{X}_\varepsilon$ and $\|u_\varepsilon\|_{\mathcal{C}^\alpha_{t_\varepsilon}}$. Combining these estimates together we obtain the required bound. $\square$

2.7.3 Estimates on the rough terms

In this section we obtain bounds on the terms involving rough integrals. As usual, we will use the notation (2.52), which in view of Remark 2.5.2 means that all the quantities involved in the definition of $\varrho_{K,\varepsilon}$ are bounded. Moreover, we will use the bounds from Section 2.6 without mentioning them. In order to make the notation shorter, let us define the quantity

$$
\begin{aligned}
\mathcal{K}_\varepsilon(t_\varepsilon) \stackrel{\text{def}}{=} \|X - X_\varepsilon\|_{\mathcal{C}^0_{t_\varepsilon}} + \|\!|\mathbf{X} - \mathbf{X}_\varepsilon|\!\|_{\alpha, t_\varepsilon} + \|\bar{v} - v_\varepsilon\|_{\mathcal{C}^\alpha_{t_\varepsilon}} \\
+ \|\bar{v} - v_\varepsilon\|_{\mathcal{C}^1_{(1-\alpha)/2, t_\varepsilon}} + \|u^0 - u^0_\varepsilon\|_{\mathcal{C}^\alpha} \; ,
\end{aligned}
\tag{2.74}
$$

where the seminorm $\|\!| \cdot |\!\|_{\alpha, t_\varepsilon}$ was introduced in (2.49).

The next lemma provides bounds on the rough integrals Z and Z_ε defined in (2.26) and (2.45) respectively.

Lemma 2.7.11. *For the constant $\alpha_\star$ defined in the beginning of Section 2.5.3 and for any $t > 0$ we have the following bound:*

$$
\|Z(t_\varepsilon)\|_{\mathcal{C}^{\alpha_\star}} \lesssim t_\varepsilon^{-\frac{\alpha_\star}{2}} \; . \tag{2.75}
$$

Moreover, we can write the difference of the two rough integrals as

$$Z - Z_\varepsilon = T_1 + T_2 \,, \tag{2.76}$$

where, for $\alpha > 0$ from the beginning of Section 2.5.3 and any $\kappa > 0$ small enough, the bounds

$$\|T_1(t_\varepsilon)\|_{\mathcal{C}^\alpha} \lesssim t_\varepsilon^{\frac{\alpha-1}{2}} \big(\mathcal{K}_\varepsilon(t_\varepsilon) + \varepsilon^{\alpha_\star - \alpha - \kappa}\big) \,, \qquad \|T_2(t_\varepsilon)\|_{\mathcal{C}^{\alpha_\star}} \lesssim \varepsilon^{3\alpha_\star - 1} t_\varepsilon^{-\frac{\alpha_\star}{2}} \,,$$

hold with $\mathcal{K}_\varepsilon$ defined in (2.74).

Proof. Since, according to Definition 2.4.2, for $s \leq t_\varepsilon$ the process $(\bar{u} - X)(s)$ belongs to the space $\mathcal{C}^1$, we can conclude that the process $Y_{ij}(s) \stackrel{\text{def}}{=} G_{ij}(\bar{u}(s))$ with the indices $i, j \in \{1, \ldots, n\}$ is controlled by the $\alpha_\star$-regular rough path $\mathbf{X}(s)$ with the rough path derivative $Y'_{ij}(s) \stackrel{\text{def}}{=} DG_{ij}(\bar{u}(s))$ and the remainder

$$\begin{aligned} R_{Y_{ij}}(s; x, y) &\stackrel{\text{def}}{=} DG_{ij}(\bar{u}(s,x))\,(\bar{v} + U)(s; x, y) \\ &+ \int_0^1 \Big(DG_{ij}\big(\lambda \bar{u}(s,y) + (1-\lambda)\bar{u}(s,x)\big) - DG_{ij}(\bar{u}(s,x))\Big)\bar{u}(s; x, y)\, d\lambda \,, \end{aligned}$$

where we use the notation $\bar{v}(s; x, y) \stackrel{\text{def}}{=} \bar{v}(s, y) - \bar{v}(s, x)$ and respectively for U and $\bar{u}$. Here, by the "rough path derivative" we mean the projection of the controlled rough path on $(\mathbb{R}^n)^*$ in Definition 2.3.2, and the "remainder" is a collection of all the processes R_Y^w from (2.15).

From the regularity assumptions for the function G and the processes $\bar{u}$ and $\bar{v}$, we obtain the bounds

$$\|Y_{ij}(s)\|_{\mathcal{C}^{\alpha_\star}} \lesssim 1 \,, \qquad \|Y'_{ij}(s)\|_{\mathcal{C}^{\alpha_\star}} \lesssim 1 \,, \qquad \|R_{Y_{ij}}(s)\|_{\mathcal{B}^{2\alpha_\star}} \lesssim s^{-\frac{\alpha_\star}{2}} \,. \tag{2.77}$$

The power of s in the last estimate comes from the estimate $\|U(s)\|_{2\alpha_\star} \lesssim s^{-\frac{\alpha_\star}{2}}$, which is a consequence of Lemma 2.6.1. Then the bound (2.75) follows from (2.20) and (2.77).

Similarly, for $s \leq t_\varepsilon$, the process $Y_{\varepsilon,ij}(s) \stackrel{\text{def}}{=} G_{ij}(u_\varepsilon(s))$ is controlled by the $\alpha_\star$-regular rough path $\mathbf{X}_\varepsilon(s)$ with the rough path derivative $Y'_{\varepsilon,ij}(s) \stackrel{\text{def}}{=} DG_{ij}(u_\varepsilon(s))$ and the respective remainder $R_{Y_{\varepsilon,ij}}(s)$, such that the following bounds hold

$$\|Y_{\varepsilon,ij}(s)\|_{\mathcal{C}^{\alpha_\star}} \lesssim 1 \,, \qquad \|Y'_{\varepsilon,ij}(s)\|_{\mathcal{C}^{\alpha_\star}} \lesssim 1 \,, \qquad \|R_{Y_{\varepsilon,ij}}(s)\|_{\mathcal{B}^{2\alpha_\star}} \lesssim s^{-\frac{\alpha_\star}{2}} \tag{2.78}$$

In order to prove the second part of this lemma, we consider the processes $\bar{u}(s)$ and $u_\varepsilon(s)$ to be of Hölder regularity $\alpha > 0$. Then they are controlled by the α-regular rough paths $\mathbf{X}(s)$ and $\mathbf{X}_\varepsilon(s)$ respectively. Hence, we can extend $G_{ij}(\bar{u}(s))$ to the process $\mathcal{G}_{ij}(s) : \mathbb{T} \to \big(T^{(p-1)}(\mathbb{R}^n)\big)^*$ which is controlled by $\mathbf{X}(s)$ as well and such that

$$\langle \mathcal{G}_{ij}(s,x), e_w \rangle \stackrel{\text{def}}{=} D^w G_{ij}(\bar{u}(s,x)) \;,$$

for each $w \in \mathcal{A}_{p-1}$. Then, as it was noticed in Section 2.5.2, for every $w \in \mathcal{A}_{p-1}$ the following expansion holds

$$\begin{aligned}&\langle \mathcal{G}_{ij}(s,y) - \mathcal{G}_{ij}(s,x), e_w \rangle \\ &\qquad = \sum_{\bar{w} \in \mathcal{A}_{p-|w|-1} \setminus \varnothing} C_{\bar{w}} \langle \mathcal{G}_{ij}(s,x), e_{\bar{w}} \otimes e_w \rangle \langle \mathbf{X}(s;x,y), e_{\bar{w}} \rangle + R^w_{\mathcal{G}_{ij}}(s;x,y) \;.\end{aligned}$$

For any word $w \in \mathcal{A}_{p-1}$, the assumptions on G and $\bar{u}$ imply $\|\langle \mathcal{G}_{ij}(s), e_w \rangle\|_{\mathcal{C}^\alpha} \lesssim 1$. Furthermore, from the argument of Section 2.5.2, it is not difficult to obtain the estimate on the remainder: $\|R^w_{\mathcal{G}_{ij}}(s)\|_{\mathcal{B}^{(p-|w|)\alpha}} \lesssim s^{\frac{\alpha_\star - 1}{2}}$. The latter bound follows from $\big|\bar{u}(s;x,y)_{\bar{w}}\big| \lesssim |y-x|^{(p-|w|)\alpha}$, for any word $\bar{w}$ such that $|\bar{w}| = p - |w|$, and

$$\begin{aligned}\big|(\bar{u} - X)(s;x,y)_{\bar{w}}\big| &\lesssim \big|(\bar{u} - X)(s;x,y)\big| \\ &\lesssim |y-x| \Big(\|\bar{v}(s)\|_{\mathcal{C}^1} + \|U(s)\|_{\mathcal{C}^1} \Big) \lesssim |y-x| \Big(1 + s^{\frac{\alpha_\star - 1}{2}} \Big) \;,\end{aligned}$$

for any word $\bar{w} \in \mathcal{A}_{p-|w|-1} \setminus \{\varnothing\}$. Here, in the last line we have used the bound

$$\|U(s)\|_{\mathcal{C}^1} \lesssim s^{\frac{\alpha_\star - 1}{2}} \Big(\|u^0\|_{\mathcal{C}^{\alpha_\star}} + \|X(0)\|_{\mathcal{C}^{\alpha_\star}} \Big) \;,$$

which follows from Lemma 2.6.1.

In the same way, $G_{ij}(u_\varepsilon(s))$ can be extended to a process $\mathcal{G}^\varepsilon_{ij}(s)$ acting from $\mathbb{T}$ to $\big(T^{(p-1)}(\mathbb{R}^n)\big)^*$ which is controlled by $\mathbf{X}_\varepsilon(s)$. We denote the remainders by $R^w_{\mathcal{G}^\varepsilon_{ij}}$. Furthermore, the corresponding bounds

$$\big\|\langle \mathcal{G}^\varepsilon_{ij}(s), e_w \rangle\big\|_{\mathcal{C}^\alpha} \lesssim 1 \;, \qquad \big\|R^w_{\mathcal{G}^\varepsilon_{ij}}(s)\big\|_{\mathcal{B}^{(p-|w|)\alpha}} \lesssim s^{\frac{\alpha_\star - 1}{2}} \;,$$

hold for any word $w \in \mathcal{A}_{p-1}$.

The following estimate follows from the regularity of the function G,

$$\begin{aligned}\big\|\langle(\mathcal{G}-\mathcal{G}^\varepsilon)_{ij}(s),e_w\rangle\big\|_{\mathcal{C}^\alpha} &\lesssim \|(\bar u-u_\varepsilon)(s)\|_{\mathcal{C}^\alpha} \\ &\lesssim \|(X-X_\varepsilon)(s)\|_{\mathcal{C}^\alpha}+\|(\bar v-v_\varepsilon)(s)\|_{\mathcal{C}^\alpha}+\|u^0-u^0_\varepsilon\|_{\mathcal{C}^\alpha}\;,\end{aligned} \tag{2.79}$$

where $w\in\mathcal{A}_{p-1}$. Furthermore, the following bound holds

$$\big|(\bar u-u_\varepsilon)(s;x,y)_{\bar w}\big|\lesssim|y-x|^{(p-|w|)\alpha}\|(\bar u-u_\varepsilon)(s)\|_{\mathcal{C}^\alpha}\;,$$

for a word $\bar w$ such that $|\bar w|=p-|w|$, and for any word $\bar w\in\mathcal{A}_{p-|w|-1}\setminus\{\varnothing\}$ one has

$$\begin{aligned}\big|\big(\bar u-X-u_\varepsilon+X_\varepsilon\big)(s;x,y)_{\bar w}\big| &\lesssim \big|\big(\bar u-X-u_\varepsilon+X_\varepsilon\big)(s;x,y)\big| \\ &\lesssim |y-x|\Big(\|(\bar v-v_\varepsilon)(s)\|_{\mathcal{C}^1}+\|(U-U_\varepsilon)(s)\|_{\mathcal{C}^1}\Big) \\ &\lesssim |y-x|s^{\frac{\alpha-1}{2}}\Big(\|\bar v-v_\varepsilon\|_{\mathcal{C}^1_{(1-\alpha)/2,s}}+\|(X-X_\varepsilon)(s)\|_{\mathcal{C}^\alpha} \\ &\qquad +\|u^0-u^0_\varepsilon\|_{\mathcal{C}^\alpha}+\varepsilon^{\alpha_\star-\alpha-\kappa}\Big)\;.\end{aligned}$$

Here, in the last line we have used the bound

$$\begin{aligned}\|(U-U_\varepsilon)(s)\|_{\mathcal{C}^1} &\lesssim \big\|S_s\big(X(0)-X_\varepsilon(0)-u^0+u^0_\varepsilon\big)\big\|_{\mathcal{C}^1} \\ &\qquad +\big\|\big(S_s-S_s^{(\varepsilon)}\big)\big(X_\varepsilon(0)-u^0_\varepsilon\big)\big\|_{\mathcal{C}^1} \\ &\lesssim s^{\frac{\alpha-1}{2}}\Big(\|(X-X_\varepsilon)(0)\|_{\mathcal{C}^\alpha}+\|u^0-u^0_\varepsilon\|_{\mathcal{C}^\alpha}+\varepsilon^{\alpha_\star-\alpha-\kappa}\Big)\;,\end{aligned}$$

for any $\kappa>0$ sufficiently small, which follows from Lemmas 2.6.1 and 2.6.2. From these bounds and Section 2.5.2 we obtain

$$\begin{aligned}\big\|\big(R^w_{\mathcal{G}_{ij}}-R^w_{\mathcal{G}^\varepsilon_{ij}}\big)(s)\big\|_{\mathcal{B}^{(p-|w|)\alpha}} &\lesssim s^{\frac{\alpha-1}{2}}\Big(\|\bar v-v_\varepsilon\|_{\mathcal{C}^1_{(1-\alpha)/2,s}}+\|X-X_\varepsilon\|_{\mathcal{C}^\alpha_s} \\ &\qquad +\|u^0-u^0_\varepsilon\|_{\mathcal{C}^\alpha}+\varepsilon^{\alpha_\star-\alpha-\kappa}\Big)\;.\end{aligned} \tag{2.80}$$

In order to prove the bounds on the terms in (2.76), we define

$$\begin{aligned}Q^\varepsilon_i(t_\varepsilon;x,y) &\stackrel{\text{def}}{=} \int_x^y G_{ij}(u_\varepsilon(t_\varepsilon,z))\,d_zX^j_\varepsilon(t_\varepsilon,z)-G_{ij}(u_\varepsilon(t_\varepsilon,x))X^j_\varepsilon(t_\varepsilon;x,y) \\ &\qquad -\sum_{w\in\mathcal{A}}D^wG_{ij}(u_\varepsilon(t_\varepsilon,x))\langle\mathbf{X}_\varepsilon(t_\varepsilon;x,y),e_w\otimes e_j\rangle\;,\end{aligned}$$

$$T_i^\varepsilon(t_\varepsilon; x, y) \stackrel{\text{def}}{=} \sum_{\substack{w \in \mathcal{A}_{p-1} \\ |w| \geq 2}} C_w \langle \mathcal{G}_{ij}^\varepsilon(t_\varepsilon, x), e_w \rangle \langle \mathbf{X}_\varepsilon(t_\varepsilon; x, y), e_w \otimes e_j \rangle \;,$$

where we have omitted as usual the sum over $j \in \{1, \dots, n\}$. From (2.19), (2.78) and Definition 2.3.1 we obtain

$$\|Q_i^\varepsilon(t_\varepsilon)\|_{\mathcal{B}^{3\alpha_\star}} \lesssim t_\varepsilon^{-\frac{\alpha_\star}{2}} \;, \qquad \|T_i^\varepsilon(t_\varepsilon)\|_{\mathcal{B}^{3\alpha_\star}} \lesssim 1 \;. \tag{2.81}$$

Next, we can rewrite $Z^i - Z^i_\varepsilon$ in the following way

$$\begin{aligned}
(Z^i - Z^i_\varepsilon)(t_\varepsilon, x) = &\left(\fint_{-\pi}^x G_i(u(t_\varepsilon, y))\, d_y X(t_\varepsilon, y) - \fint_{-\pi}^x G_i(u_\varepsilon(t_\varepsilon, y))\, d_y X_\varepsilon(t_\varepsilon, y) \right) \\
&+ \int_{\mathbb{R}} \fint_{-\pi}^{-\pi+\varepsilon z} \frac{\varepsilon z - \pi - y}{\varepsilon} G_i(u_\varepsilon(t_\varepsilon, y))\, d_y X_\varepsilon(t_\varepsilon, y)\, \mu(dz) \\
&+ \int_{\mathbb{R}} \fint_{x}^{x+\varepsilon z} \frac{y - \varepsilon z - x}{\varepsilon} G_i(u_\varepsilon(t_\varepsilon, y))\, d_y X_\varepsilon(t_\varepsilon, y)\, \mu(dz) \\
&- \int_{\mathbb{R}} \int_{-\pi}^{x} \frac{Q_i^\varepsilon(t_\varepsilon; y, y + \varepsilon z)}{\varepsilon}\, dy\, \mu(dz) \\
&+ \int_{\mathbb{R}} \int_{-\pi}^{x} \frac{T_i^\varepsilon(t_\varepsilon; y, y + \varepsilon z)}{\varepsilon}\, dy\, \mu(dz) \\
\stackrel{\text{def}}{=} &\sum_{1 \leq j \leq 5} I_j(t_\varepsilon, x) \;.
\end{aligned}$$

Here, we have used the Fubini-type result proved in [HW13, Lem. 2.10].

In order to bound I_1 we apply (2.21) and use the bounds (2.79), (2.80),

$$\|I_1(t_\varepsilon)\|_{\mathcal{C}^\alpha} \lesssim t_\varepsilon^{\frac{\alpha-1}{2}} \left(\mathcal{K}_\varepsilon(t_\varepsilon) + \varepsilon^{\alpha_\star - \alpha - \kappa} \right) \;,$$

where $\mathcal{K}_\varepsilon$ is defined in (2.74). Furthermore, it follows from (2.81) that

$$\|I_4(t_\varepsilon)\|_{\mathcal{C}^1} \lesssim \varepsilon^{3\alpha_\star - 1} \|Q_i^\varepsilon(t_\varepsilon)\|_{\mathcal{B}^{3\alpha_\star}} \int_{\mathbb{R}} |z|^{3\alpha_\star} \mu(dz) \lesssim \varepsilon^{3\alpha_\star - 1} t_\varepsilon^{-\frac{\alpha_\star}{2}} \;.$$

In the same way from the second bound in (2.81) we derive

$$\|I_5(t_\varepsilon)\|_{\mathcal{C}^1} \lesssim \varepsilon^{3\alpha_\star - 1} \|T_i^\varepsilon(t_\varepsilon)\|_{\mathcal{B}^{3\alpha_\star}} \int_{\mathbb{R}} |z|^{3\alpha_\star} \mu(dz) \lesssim \varepsilon^{3\alpha_\star - 1} \;.$$

In order to bound the third integral I_3, let us define

$$\begin{aligned} u_{x,z,\varepsilon}(t_\varepsilon, y) &\stackrel{\text{def}}{=} u_\varepsilon(t_\varepsilon, \varepsilon y - \varepsilon z - x) \;, \\ \mathbf{X}_{x,z,\varepsilon}(t_\varepsilon; y, \bar{y}) &\stackrel{\text{def}}{=} \mathbf{X}_\varepsilon(t_\varepsilon; \varepsilon y - \varepsilon z - x, \varepsilon\bar{y} - \varepsilon z - x) \;. \end{aligned}$$

Then we can perform the change of variables $\bar{y} = (y - \varepsilon z - x)/\varepsilon$ in the integral I_3 and obtain

$$I_3 = \int_{\mathbb{R}} \fint_{-z}^{0} Y_{x,z,\varepsilon}(t_\varepsilon, \bar{y})\, d_{\bar{y}} X_{x,z,\varepsilon}(t_\varepsilon, \bar{y})\, \mu(dz) \;,$$

where $X_{x,z,\varepsilon}(t_\varepsilon, \bar{y}) - X_{x,z,\varepsilon}(t_\varepsilon, y)$ is the projection of $\mathbf{X}_{x,z,\varepsilon}(t_\varepsilon; y, \bar{y})$ onto $\mathbb{R}^n$ and

$$Y_{x,z,\varepsilon}(t_\varepsilon, \bar{y}) \stackrel{\text{def}}{=} \bar{y}\, G_i(u_{x,z,\varepsilon}(t_\varepsilon, \bar{y})) \;.$$

Taking into account the a priori bounds on u_ε, we obtain from [Hai11, Lem. 2.2] that $Y_{x,z,\varepsilon}(t_\varepsilon)$ is controlled by $\mathbf{X}_{x,z,\varepsilon}(t_\varepsilon)$ with the rough path derivative

$$Y'_{x,z,\varepsilon}(t_\varepsilon, \bar{y}) \stackrel{\text{def}}{=} \bar{y}\, DG_i(u_{x,z,\varepsilon}(t_\varepsilon, \bar{y}))$$

and the remainder $R_{Y_{x,z,\varepsilon}}(t_\varepsilon)$ such that

$$\|Y_{x,z,\varepsilon}(t_\varepsilon)\|_{\mathcal{C}^{\alpha_\star}} \lesssim 1 \;, \qquad \|Y'_{x,z,\varepsilon}(t_\varepsilon)\|_{\mathcal{C}^{\alpha_\star}} \lesssim 1 \;, \qquad \|R_{Y_{x,z,\varepsilon}}(t_\varepsilon)\|_{\mathcal{B}^{2\alpha_\star}} \lesssim t_\varepsilon^{-\frac{\alpha_\star}{2}} \;.$$

Hence, using Proposition 2.3.3 and the simple estimate

$$\|\!|\mathbf{X}_{x,z,\varepsilon}(t_\varepsilon)|\!\|_{\alpha_\star} \le \varepsilon^{\alpha_\star} \|\!|\mathbf{X}_\varepsilon(t_\varepsilon)|\!\|_{\alpha_\star} \;,$$

we obtain the following bound:

$$\begin{aligned} \|I_3(t_\varepsilon)\|_{\mathcal{C}^{\alpha_\star}} &\le \int_{\mathbb{R}} \Big\| \fint_{\cdot} Y_{x,z,\varepsilon}(t_\varepsilon, \bar{y})\, d_{\bar{y}} X_{x,z,\varepsilon}(t_\varepsilon, \bar{y}) \Big\|_{\mathcal{C}^{\alpha_\star}} |z|^{\alpha_\star}\, \mu(dz) \\ &\lesssim \int_{\mathbb{R}} \|\!|\mathbf{X}_{x,z,\varepsilon}(t_\varepsilon)|\!\|_{\alpha_\star} \Big(\|Y_{x,z,\varepsilon}(t_\varepsilon)\|_{\mathcal{C}^{\alpha_\star}} + \|Y'_{x,z,\varepsilon}(t_\varepsilon)\|_{\mathcal{C}^{\alpha_\star}} \\ &\qquad + \|R_{Y_{x,z,\varepsilon}}(t_\varepsilon)\|_{\mathcal{B}^{2\alpha_\star}} \Big) |z|^{\alpha_\star}\, \mu(dz) \\ &\lesssim \varepsilon^{\alpha_\star} t_\varepsilon^{-\frac{\alpha_\star}{2}} \;. \end{aligned}$$

Here we have also used the bound on the $\alpha_\star$-th moment of the measure μ. Similarly, we can obtain the bound $\|I_2(t_\varepsilon)\|_{\mathcal{C}^{\alpha_\star}} \lesssim \varepsilon^{\alpha_\star} t_\varepsilon^{-\frac{\alpha_\star}{2}}$.

Setting now $T_1 = I_1$ and $T_2 = I_2 + I_3 + I_4 + I_5$, we obtain the claim. $\square$

In the following proposition we prove a bound on the terms $\mathbf{Z}^{\bar v}$ and $\mathbf{Z}_\varepsilon^{v_\varepsilon}$ defined in (2.28) and (2.47) respectively.

Proposition 2.7.12. *For any $\gamma \in (0,1]$, any $t > 0$ and any $\kappa > 0$ small enough one has the bound*

$$\big\|\big(\mathbf{Z}^{\bar v} - \mathbf{Z}_\varepsilon^{v_\varepsilon}\big)(t_\varepsilon)\big\|_{\mathcal{C}^\gamma} \lesssim t_\varepsilon^{\alpha - \frac{1}{2}(\gamma+\kappa)} \big(\mathcal{K}_\varepsilon(t_\varepsilon) + \varepsilon^{\alpha_\star - \alpha - \kappa}\big) \,,$$

where $\mathcal{K}_\varepsilon$ is defined in (2.74) *and the constants α and $\alpha_\star$ are as in Lemma 2.7.11.*

Proof. We can rewrite $\mathbf{Z}^{\bar v} - \mathbf{Z}_\varepsilon^{v_\varepsilon}$ in the following way

$$\begin{aligned}\big(\mathbf{Z}^{\bar v} - \mathbf{Z}_\varepsilon^{v_\varepsilon}\big)(t_\varepsilon) &= \int_0^{t_\varepsilon} \partial_x \big(S_{t_\varepsilon - s} - S^{(\varepsilon)}_{t_\varepsilon - s}\big) Z(s)\, ds + \int_0^{t_\varepsilon} \partial_x S^{(\varepsilon)}_{t_\varepsilon - s} (Z - Z_\varepsilon)(s)\, ds \\ &\stackrel{\text{def}}{=} J_1 + J_2 \,.\end{aligned}$$

Using Lemmas 2.7.11 and 2.6.2 with $\lambda = \alpha_\star - \alpha - \kappa$ we obtain, for any $\kappa > 0$ small enough,

$$\begin{aligned}\|J_1\|_{\mathcal{C}^\gamma} &\lesssim \int_0^{t_\varepsilon} \big\|S_{t_\varepsilon - s} - S^{(\varepsilon)}_{t_\varepsilon - s}\big\|_{\mathcal{C}^{\alpha_\star} \to \mathcal{C}^{1+\gamma}} \|Z(s)\|_{\mathcal{C}^{\alpha_\star}}\, ds \\ &\lesssim \varepsilon^{\alpha_\star - \alpha - \kappa} \int_0^{t_\varepsilon} (t_\varepsilon - s)^{-\frac{1}{2}(1+\gamma-\alpha)} s^{-\frac{\alpha_\star}{2}}\, ds \\ &\lesssim t_\varepsilon^{\frac{1}{2}(1-\gamma+\alpha-\alpha_\star)} \varepsilon^{\alpha_\star - \alpha - \kappa} \,.\end{aligned}$$

The second term can be estimated using Lemma 2.6.4 and (2.76) by

$$\begin{aligned}\|J_2\|_{\mathcal{C}^\gamma} &\lesssim \int_0^{t_\varepsilon} \big\|S^{(\varepsilon)}_{t_\varepsilon - s}\big\|_{\mathcal{C}^{\alpha} \to \mathcal{C}^{1+\gamma}} \|T_1(s)\|_{\mathcal{C}^\alpha} ds + \int_0^{t_\varepsilon} \big\|S^{(\varepsilon)}_{t_\varepsilon - s}\big\|_{\mathcal{C}^{\alpha_\star} \to \mathcal{C}^{1+\gamma}} \|T_2(s)\|_{\mathcal{C}^{\alpha_\star}} ds \\ &\lesssim \int_0^{t_\varepsilon} (t_\varepsilon - s)^{-\frac{1}{2}(1+\gamma-\alpha+\kappa)} s^{\frac{\alpha-1}{2}} \big(\mathcal{K}_\varepsilon(s) + \varepsilon^{\alpha_\star - \alpha - \kappa}\big)\, ds \\ &\qquad + \varepsilon^{3\alpha_\star - 1} \int_0^{t_\varepsilon} (t_\varepsilon - s)^{-\frac{1}{2}(1+\gamma-\alpha_\star+\kappa)} s^{-\frac{\alpha_\star}{2}}\, ds \\ &\lesssim t_\varepsilon^{\alpha - \frac{1}{2}(\gamma+\kappa)} \big(\mathcal{K}_\varepsilon(t_\varepsilon) + \varepsilon^{\alpha_\star - \alpha - \kappa}\big) + \varepsilon^{3\alpha_\star - 1} t_\varepsilon^{\frac{1}{2}(1-\gamma-\kappa)} \,.\end{aligned}$$

Combining these bounds together we obtain the required estimate. $\square$

2.8 Proof of the convergence result

With the results from the previous sections at hand, we can now prove Theorem 2.2.7.

Proof of Theorem 2.2.7. For $\alpha \in \left(0, \frac{1}{2}\right)$ we define $\alpha_\star \stackrel{\text{def}}{=} \frac{1}{2} - \alpha$ as in the beginning of Section 2.5.3. From the derivation of the bounds below we will see how small the value of α must be. To make the notation shorter, we introduce the following norm

$$\| \cdot \|_{\alpha,t} \stackrel{\text{def}}{=} \| \cdot \|_{\mathcal{C}^\alpha_t} + \| \cdot \|_{\mathcal{C}^1_{(1-\alpha)/2,t}} .$$

Then, using the notation (2.52), we obtain from (2.27) and (2.46) the bound

$$\begin{aligned} \|\bar{v} - v_\varepsilon\|_{\alpha,t_\varepsilon} \leq \big\|\mathbf{G}^{\bar{v}} - \mathbf{G}^{v_\varepsilon}_\varepsilon\big\|_{\alpha,t_\varepsilon} + \big\|\mathbf{F}^{\bar{v}} - \mathbf{F}^{v_\varepsilon}_\varepsilon\big\|_{\alpha,t_\varepsilon} + \big\|\mathbf{H}^{\bar{v}} - \mathbf{H}^{v_\varepsilon}_\varepsilon\big\|_{\alpha,t_\varepsilon} \\ + \big\|\bar{\mathbf{H}}^{v_\varepsilon}_\varepsilon\big\|_{\alpha,t_\varepsilon} + \big\|\mathbf{Z}^{\bar{v}} - \mathbf{Z}^{v_\varepsilon}_\varepsilon\big\|_{\alpha,t_\varepsilon} . \end{aligned} \tag{2.82}$$

In what follows, we consider only time periods $t_\varepsilon < 1$, for larger times the claim can be obtained by iterations in the same way how it was done in [HMW14, Thm. 2.4].

To find a bound on the first term in (2.82) we use the results of Section 2.7.1. Applying Proposition 2.7.1 with a small constant $\kappa = \alpha$ we get

$$\big\|\mathbf{G}^{\bar{v}} - \mathbf{G}^{v_\varepsilon}_\varepsilon\big\|_{\alpha,t_\varepsilon} \lesssim t_\varepsilon^{\frac{1}{2}} \|\bar{v} - v_\varepsilon\|_{\alpha,t_\varepsilon} + \|X - X_\varepsilon\|_{\mathcal{C}^\alpha_{t_\varepsilon}} + \|u^0 - u^0_\varepsilon\|_{\mathcal{C}^\alpha} + \varepsilon^{\alpha_\star - \alpha} .$$

In order to bound the second term in (2.82), we use Proposition 2.7.2 with $\kappa = \alpha$,

$$\big\|\mathbf{F}^{\bar{v}} - \mathbf{F}^{v_\varepsilon}_\varepsilon\big\|_{\alpha,t_\varepsilon} \lesssim t_\varepsilon^{\frac{1-\alpha}{2}} \|\bar{v} - v_\varepsilon\|_{\mathcal{C}^0_{t_\varepsilon}} + \|X - X_\varepsilon\|_{\mathcal{C}^0_{t_\varepsilon}} + \|u^0 - u^0_\varepsilon\|_{\mathcal{C}^0} + \varepsilon^{\alpha_\star - \alpha} .$$

Applying Proposition 2.7.10 with the same values of κ, we bound the expectation of the third term in (2.82) by

$$\mathbb{E}\big\|\mathbf{H}^{\bar{v}} - \mathbf{H}^{v_\varepsilon}_\varepsilon\big\|_{\alpha,t_\varepsilon} \lesssim T^{\frac{1-\alpha}{2}} \mathbb{E}\|\bar{v} - v_\varepsilon\|_{\mathcal{C}^0_{t_\varepsilon}} + \mathbb{E}\|X - X_\varepsilon\|_{\mathcal{C}^0_{t_\varepsilon}} + \mathbb{E}\|u^0 - u^0_\varepsilon\|_{\mathcal{C}^0} + \varepsilon^{\alpha_\star - \alpha},$$

where $T > 0$ is as in (2.51). A bound on the fourth term in (2.82) is a straightforward application of Proposition 2.7.4, i.e.

$$\big\|\bar{\mathbf{H}}^{v_\varepsilon}_\varepsilon\big\|_{\mathcal{C}^\alpha_{t_\varepsilon}} + \big\|\bar{\mathbf{H}}^{v_\varepsilon}_\varepsilon\big\|_{\mathcal{C}^1_{t_\varepsilon}} \lesssim \varepsilon^{3\alpha_\star - 1} .$$

Using Proposition 2.7.12 with the small parameter $\kappa = \frac{\alpha}{2}$ we can bound the last term in (2.82) by

$$\left\|\mathbf{Z}^{\bar v} - \mathbf{Z}_\varepsilon^{v_\varepsilon}\right\|_{\alpha,t_\varepsilon} \lesssim t_\varepsilon^{\frac{\alpha}{4}} \mathcal{K}_\varepsilon(t_\varepsilon) + \varepsilon^{\alpha_\star - \frac{3\alpha}{2}} ,$$

where $\mathcal{K}_\varepsilon$ is defined in (2.74).

Combining all the bounds from above together we obtain

$$\begin{aligned}\mathbb{E}\|\bar v - v_\varepsilon\|_{\alpha,t_\varepsilon} \lesssim T^{\frac{\alpha}{4}} \mathbb{E}\|\bar v - v_\varepsilon\|_{\alpha,t_\varepsilon} + \mathbb{E}\|u^0 - u^0_\varepsilon\|_{\mathcal{C}^\alpha} + \mathbb{E}\|X - X_\varepsilon\|_{\mathcal{C}^\alpha_{t_\varepsilon}} \\ + \mathbb{E}\|\!|\mathbf{X} - \mathbf{X}_\varepsilon|\!\|_{\alpha,t_\varepsilon} + \varepsilon^{\frac{1}{2}-3\alpha} .\end{aligned} \tag{2.83}$$

By Lemma 2.5.1 we can bound the norms of the controlling processes,

$$\mathbb{E}\|X - X_\varepsilon\|_{\mathcal{C}^\alpha_{t_\varepsilon}} + \mathbb{E}\|\!|\mathbf{X} - \mathbf{X}_\varepsilon|\!\|_{\alpha,t_\varepsilon} \lesssim \varepsilon^{\frac{1}{2}-2\alpha} .$$

Furthermore, by choosing T in (2.51) small enough we can absorb the first term on the right-hand side of (2.83) into the left-hand side and obtain

$$\mathbb{E}\|\bar v - v_\varepsilon\|_{\alpha,t_\varepsilon} \le C\Big(\mathbb{E}\|u^0 - u^0_\varepsilon\|_{\mathcal{C}^\alpha} + \varepsilon^{\frac{1}{2}-3\alpha}\Big) . \tag{2.84}$$

Thus, from the definition of $\bar u$ via $\bar v$ and (2.84) we conclude

$$\begin{aligned}\mathbb{E}\|\bar u - u_\varepsilon\|_{\mathcal{C}^\alpha_{t_\varepsilon}} &\le \mathbb{E}\|\bar v - v_\varepsilon\|_{\mathcal{C}^\alpha_{t_\varepsilon}} + \mathbb{E}\|X - X_\varepsilon\|_{\mathcal{C}^\alpha_{t_\varepsilon}} + \mathbb{E}\|U - U_\varepsilon\|_{\mathcal{C}^\alpha_{t_\varepsilon}} \\ &\le C\mathbb{E}\|u^0 - u^0_\varepsilon\|_{\mathcal{C}^\alpha} + \varepsilon^{\frac{1}{2}-3\alpha} .\end{aligned} \tag{2.85}$$

Here, we have also used Lemma 2.5.1 and the bound

$$\begin{aligned}\|(U - U_\varepsilon)(t)\|_{\mathcal{C}^\alpha} \lesssim \|u^0 - u^0_\varepsilon\|_{\mathcal{C}^\alpha} + \|(X - X_\varepsilon)(0)\|_{\mathcal{C}^\alpha} \\ + \varepsilon^{\alpha_\star - 2\alpha}\Big(\|u^0_\varepsilon\|_{\mathcal{C}^{\alpha_\star}} + \|X_\varepsilon(0)\|_{\mathcal{C}^{\alpha_\star}}\Big) ,\end{aligned}$$

which can be derived similarly to (2.56).

What we have to show now is that for any $\tilde\gamma < \alpha_\star$ we have

$$\lim_{K\uparrow\infty}\lim_{\varepsilon\to 0} \mathbb{P}\Big[\|\bar u - \ u_\varepsilon\|_{\mathcal{C}^0_{T^*_K}} \ge \varepsilon^{\tilde\gamma}\Big] = 0 ,$$

where we have used the stopping time T^*_K defined in (2.11). Then the sequence of stopping times T_ε in Theorem 2.2.7 can be chosen as a suitable diagonal sequence.

To this end, we fix a constant $\bar{K} > K$ and get

$$\mathbb{P}\Big[\|\bar{u} - u_\varepsilon\|_{\mathcal{C}^0_{T_K^*}} \geq \varepsilon^{\tilde{\gamma}}\Big] \leq \mathbb{P}\Big[\|\bar{u} - u_\varepsilon\|_{\mathcal{C}^0_{\varrho_{\bar{K},\varepsilon}}} \geq \varepsilon^{\tilde{\gamma}}\Big] + \mathbb{P}\Big[\varrho_{\bar{K},\varepsilon} < \sigma_{\bar{K}}\Big] + \mathbb{P}\Big[\sigma_{\bar{K}} < T_K^*\Big] . \tag{2.86}$$

By the Chebyshev inequality and (2.85) we obtain a bound on the first term in (2.86),

$$\mathbb{P}\Big[\|\bar{u} - u_\varepsilon\|_{\mathcal{C}^0_{\varrho_{\bar{K},\varepsilon}}} \geq \varepsilon^{\tilde{\gamma}}\Big] \leq \frac{1}{\varepsilon^{\tilde{\gamma}}}\mathbb{E}\Big[\|\bar{u} - u_\varepsilon\|_{\mathcal{C}^0_{\varrho_{\bar{K},\varepsilon}}}\Big] \lesssim \varepsilon^{\alpha_\star - \tilde{\gamma}} ,$$

which vanishes as $\varepsilon \to 0$.

Recalling furthermore the definitions of the stopping times (2.50) and (2.51) we get

$$\mathbb{P}\Big[\varrho_{\bar{K},\varepsilon} < \sigma_{\bar{K}}\Big] \leq \mathbb{P}\Big[\|X - X_\varepsilon\|_{\mathcal{C}^{\alpha_\star}_{\varrho_{\bar{K},\varepsilon}}} \geq 1 \text{ or } \|\mathbf{X} - \mathbf{X}_\varepsilon\|_{\mathcal{B}^{2\alpha_\star}_{\varrho_{\bar{K},\varepsilon}}} \geq 1, \text{ or } \|\mathcal{H}_\varepsilon\|_{\mathcal{C}^{-\frac{1}{2}+\frac{\alpha}{3}}_{\varrho_{\bar{K},\varepsilon}}} \geq 1, \text{ or } \|\bar{v} - v_\varepsilon\|_{\mathcal{C}^{\alpha}_{\varrho_{\bar{K},\varepsilon}}} \geq 1, \text{ or } \|\bar{v} - v_\varepsilon\|_{\mathcal{C}^1_{(1-\alpha)/2,\varrho_{\bar{K},\varepsilon}}} \geq 1\Big] .$$

According to Lemma 2.5.1, the probabilities of the events involving X, $\mathbf{X}$ and their approximations vanish as $\varepsilon \to 0$. By Proposition 2.7.7 the probability of the event involving $\mathcal{H}_\varepsilon$ goes to 0 as $\varepsilon \to 0$. According to (2.84), the probability of the event involving $\bar{v}$ and v_ε vanishes as well.

Finally, for the last term in (2.86) we have

$$\mathbb{P}\big[\sigma_{\bar{K}} < T_K^*\big] \leq \mathbb{P}\Big[\|X\|_{\mathcal{C}^{\alpha_\star}_{T_K^*}} \geq \bar{K} \text{ or } \|\mathbf{X}\|_{\mathcal{B}^{2\alpha_\star}_{T_K^*}} \geq \bar{K}, \text{ or } \|\bar{v}\|_{\mathcal{C}^{1+\alpha_\star}_{\alpha_\star/2,T_K^*}} \geq \bar{K}, \text{ or } \|\bar{v}\|_{\mathcal{C}^1_{T_K^*}} \geq \bar{K}, \text{ or } \|v_\varepsilon\|_{\mathcal{C}^1_{T_K^*}} \geq \bar{K}\Big] .$$

The probabilities of the events $\|X\|_{\mathcal{C}^{\alpha_\star}_{T_K^*}} \geq \bar{K}$ and $\|\mathbf{X}\|_{\mathcal{B}^{2\alpha_\star}_{T_K^*}} \geq \bar{K}$ converge to 0 as $\bar{K} \uparrow \infty$, what follows from Lemma 2.5.1. Convergence to 0 of the probability of the events involving $\bar{v}$ and v_ε follows from the well-posedness of the equation (2.46) on the time interval $[0, T_K^*]$ and Remark 2.4.5. $\square$

Chapter 3

Regularity structures and solutions to rough stochastic PDEs

3.1 Introduction

The aim of this chapter is to describe the framework of regularity structures which allows to solve the parabolic stochastic PDEs of the form

$$\partial_t u = Au + F(u, \xi) \;, \tag{3.1}$$

where A is an elliptic differential operator, ξ is a rough noise, and F is a non-linear function in u which is affine in ξ. The main assumption on the equation (3.1) is *local subcriticality*, which roughly speaking means that if we rescale the equation in a way that keeps both the linear part and the noise ξ invariant, then at small scales the nonlinear terms formally disappear.

In this chapter we describe the algebraic and analytic steps of the strategy described in Section 1.2, i.e. we describe a kind of "abstract Taylor expansions" of the solution around any point in space-time and solve the equation on the "abstract level" as a fixed point in a space of such expansions. Moreover, we define a "model" and a "reconstruction map", which relate the abstract expansions to concrete functions or distributions. In contrast to the original theory [Hai14], in which the author considered all objects as distributions in space-time, we would like to work in spaces of functions in time with the values in spaces of distributions. This idea requires some modification of the original theory, in particular we introduce a new definition of "models" and we change the "abstract integration" operation [Hai14, Sec. 5] and the corresponding Schauder-type estimates.

A particular example prototypical for the class of equations we are interested in is the dynamical Φ^4 model in dimension 3, which can be formally described by the equation

$$\partial_t \Phi = \Delta\Phi + \infty\,\Phi - \Phi^3 + \xi \;, \qquad \Phi(0, \cdot) = \Phi_0(\cdot) \;, \tag{3.2}$$

on the torus $\mathbb{T}^3 \stackrel{\text{def}}{=} (\mathbb{R}/\mathbb{Z})^3$ and for $t \geq 0$, where Δ is the Laplace operator on $\mathbb{T}^3$, Φ_0 is some initial data, and ξ is the space-time white noise over $L^2(\mathbb{R} \times \mathbb{T}^3)$, see [PW81].

Here, ∞ denotes an "infinite constant": (3.2) should be interpreted as the limit of solutions to the equation obtained by mollifying ξ and replacing ∞ by a constant which diverges in a suitable way as the mollifier tends to the identity. It was shown in [Hai14] that this limit exists and is independent of the choice of mollifier. The reason for the appearance of this infinite constant is that solutions are random Schwartz distributions (this is already the case for the linear equation, see [DPZ14]), so that their third power is undefined. The above notation also correctly suggests that solutions to (3.2) still depend on one parameter, namely the "finite part" of the infinite constant, but this will not be relevant here and we consider this as being fixed from now on.

In two spatial dimensions, a solution theory for (3.2) was given in [AR91, DPD03], see also [JLM85] for earlier work on a closely related model. In three dimensions, alternative approaches to (3.2) were recently obtained in [CC13] (via paracontrolled distributions, see [GIP15] for the development of that approach), and in [Kup15] (via renormalisation group techniques à la Wilson).

Structure of the chapter

This chapter has the following structure: In Section 3.2 we introduce regularity structures and inhomogeneous models (i.e. models which are functions in the time variable). Furthermore, we prove here the key results of the theory in our present framework, namely the reconstruction theorem and the Schauder estimates. In Section 3.3 we provide a solution theory for a general class of parabolic stochastic PDEs.

3.1.1 Notations and conventions

Throughout this chapter, we will work in $\mathbb{R}^{d+1}$ where d is the dimension of space and 1 is the dimension of time. Moreover, we consider the time-space scaling $\mathfrak{s} = (\mathfrak{s}_0, 1, \ldots, 1)$ of $\mathbb{R}^{d+1}$, where $\mathfrak{s}_0 > 0$ is an integer time scaling and $\mathfrak{s}_i = 1$, for $i = 1, \ldots, d$, is the scaling in each spatial direction. We set $|\mathfrak{s}| \stackrel{\text{def}}{=} \sum_{i=0}^{d} \mathfrak{s}_i$, denote by $|x|_\infty$ the ℓ^∞-norm of a point $x \in \mathbb{R}^d$, and define $\|z\|_\mathfrak{s} \stackrel{\text{def}}{=} |t|^{1/\mathfrak{s}_0} \vee |x|_\infty$ to be the $\mathfrak{s}$-scaled ℓ^∞-norm of $z = (t, x) \in \mathbb{R}^{d+1}$. For a multiindex $k \in \mathbb{N}^{d+1}$ we define $|k|_\mathfrak{s} \stackrel{\text{def}}{=} \sum_{i=0}^{d} \mathfrak{s}_i k_i$, and for $k \in \mathbb{N}^d$ with the scaling $(1, \ldots, 1)$ we denote the respective norm by $|k|$. (Our natural numbers $\mathbb{N}$ include 0.)

For $r > 0$, we denote by $\mathcal{C}^r(\mathbb{R}^d)$ the usual Hölder space on $\mathbb{R}^d$, by $\mathcal{C}^r_0(\mathbb{R}^d)$ we denote the space of compactly supported $\mathcal{C}^r$-functions and by $\mathcal{B}^r_0(\mathbb{R}^d)$ we denote the set of $\mathcal{C}^r$-functions, compactly supported in $B(0,1)$ (the unit ball centered at the origin) and with the $\mathcal{C}^r$-norm bounded by 1.

For $\varphi \in \mathcal{B}_0^r(\mathbb{R}^d)$, $\lambda > 0$ and $x, y \in \mathbb{R}^d$ we define $\varphi_x^\lambda(y) \stackrel{\text{def}}{=} \lambda^{-d}\varphi(\lambda^{-1}(y-x))$. For $\alpha < 0$, we define the space $\mathcal{C}^\alpha(\mathbb{R}^d)$ to consist of $\zeta \in \mathcal{S}'(\mathbb{R}^d)$, the space of tempered distributions, belonging to the dual space of the space of $\mathcal{C}_0^r$-functions, with $r > -\lfloor \alpha \rfloor$, and such that

$$\|\zeta\|_{\mathcal{C}^\alpha} \stackrel{\text{def}}{=} \sup_{\varphi \in \mathcal{B}_0^r} \sup_{x \in \mathbb{R}^d} \sup_{\lambda \in (0,1]} \lambda^{-\alpha} |\langle \zeta, \varphi_x^\lambda \rangle| < \infty \;. \tag{3.3}$$

Furthermore, for a function $\mathbb{R} \ni t \mapsto \zeta_t$ we define the operator $\delta^{s,t}$ by

$$\delta^{s,t}\zeta \stackrel{\text{def}}{=} \zeta_t - \zeta_s \;, \tag{3.4}$$

and for $\delta > 0$, $\eta \leq 0$ and $T > 0$, we define the space $\mathcal{C}_\eta^{\delta,\alpha}\big([0,T], \mathbb{R}^d\big)$ to consist of the functions $(0,T] \ni t \mapsto \zeta_t \in \mathcal{C}^\alpha(\mathbb{R}^d)$, such that the following norm is finite

$$\|\zeta\|_{\mathcal{C}^{\delta,\alpha}_{\eta,T}} \stackrel{\text{def}}{=} \sup_{t \in (0,T]} |t|_0^{-\eta} \|\zeta_t\|_{\mathcal{C}^\alpha} + \sup_{s \neq t \in (0,T]} |t,s|_0^{-\eta} \frac{\|\delta^{s,t}\zeta\|_{\mathcal{C}^{\alpha-\delta}}}{|t-s|^{\delta/\mathfrak{s}_0}} \;, \tag{3.5}$$

where $|t|_0 \stackrel{\text{def}}{=} |t|^{1/\mathfrak{s}_0} \wedge 1$ and $|t,s|_0 \stackrel{\text{def}}{=} |t|_0 \wedge |s|_0$. The space $\mathcal{C}_\eta^{0,\alpha}\big([0,T], \mathbb{R}^d\big)$ contains the function ζ as above which are continuous in time and is equipped with the norm defined by the first term in (3.5).

Sometimes we will need to work with space-time distributions with scaling $\mathfrak{s}$. In order to describe their regularities, we define, for a test function φ on $\mathbb{R}^{d+1}$, for $\lambda > 0$ and $z, \bar{z} \in \mathbb{R}^{d+1}$,

$$\varphi_z^{\lambda,\mathfrak{s}}(\bar{z}) \stackrel{\text{def}}{=} \lambda^{-|\mathfrak{s}|}\varphi\big(\lambda^{-s_0}(\bar{z}_0 - z_0), \lambda^{-1}(\bar{z}_1 - z_1), \ldots, \lambda^{-1}(\bar{z}_d - z_d)\big) \;, \tag{3.6}$$

and we define the space $\mathcal{C}_\mathfrak{s}^\alpha(\mathbb{R}^{d+1})$ similarly to $\mathcal{C}^\alpha(\mathbb{R}^d)$, but using the scaled functions (3.6) in (3.3).

Finally, we denote by $\star$ the convolution on $\mathbb{R}^{d+1}$, and by $x \lesssim y$ we mean that there exists a constant C independent of the relevant quantities such that $x \leq Cy$.

3.2 Regularity structures

In this section we recall the definition of a regularity structure and we introduce the inhomogeneous models used in this work, which are maps from $\mathbb{R}$ (the time coordinate) to the usual space of models as in [Hai14, Def. 2.17], endowed with a norm enforcing some amount of time regularity. Furthermore, we define inhomogeneous modelled distributions and prove the respective reconstruction theorem and Schauder estimates. Throughout this section, we work with the scaling $\mathfrak{s} = (\mathfrak{s}_0, 1, \ldots, 1)$ of $\mathbb{R}^{d+1}$, but all our results can easily be generalised to any non-Euclidean scaling in space, similarly to [Hai14].

3.2.1 Regularity structures and inhomogeneous models

The purpose of regularity structures, introduced in [Hai14] and motivated by [Lyo98, Gub04], is to generalise Taylor expansions using essentially arbitrary functions or distributions instead of polynomials. The precise definition is as follows.

Definition 3.2.1. A *regularity structure* $\mathscr{T} = (\mathcal{T}, \mathcal{G})$ consists of two objects:

- A *model space* $\mathcal{T}$, which is a graded vector space $\mathcal{T} = \bigoplus_{\alpha \in \mathcal{A}} \mathcal{T}_\alpha$, where each $\mathcal{T}_\alpha$ is a (finite dimensional in our case) Banach space and $\mathcal{A} \subset \mathbb{R}$ is a finite set of "homogeneities".
- A *structure group* $\mathcal{G}$ of linear transformations of $\mathcal{T}$, such that for every $\Gamma \in \mathcal{G}$, every $\alpha \in \mathcal{A}$ and every $\tau \in \mathcal{T}_\alpha$ one has $\Gamma\tau - \tau \in \mathcal{T}_{<\alpha}$, with $\mathcal{T}_{<\alpha} \stackrel{\text{def}}{=} \bigoplus_{\beta < \alpha} \mathcal{T}_\beta$.

In [Hai14, Def. 2.1], the set $\mathcal{A}$ was only assumed to be locally finite and bounded from below. Our assumption is more strict, but does not influence anything in the analysis of the equations we consider. In addition, our definition rules out the ambiguity of topologies on $\mathcal{T}$.

Remark 3.2.2. One of the simplest non-trivial examples of a regularity structure is given by the "abstract polynomials" in $d+1$ indeterminates X_i, with $i = 0, \ldots, d$. The set $\mathcal{A}$ in this case consists of the values $\alpha \in \mathbb{N}$ such that $\alpha \leq r$, for some $r < \infty$ and, for each $\alpha \in \mathcal{A}$, the space $\mathcal{T}_\alpha$ contains all monomials in the X_i of scaled degree α. The structure group $\mathcal{G}_{\text{poly}}$ is then simply the group of translations in $\mathbb{R}^{d+1}$ acting on X^k by $h \mapsto (X - h)^k$.

We now fix $r > 0$ to be sufficiently large and denote by $\mathcal{T}_{\mathrm{poly}}$ the space of such polynomials of scaled degree r and by $\mathcal{F}_{\mathrm{poly}}$ the set $\{X^k : |k|_{\mathfrak{s}} \leq r\}$. We will only ever consider regularity structures containing $\mathcal{T}_{\mathrm{poly}}$ as a subspace. In particular, we always assume that there's a natural morphism $\mathcal{G} \to \mathcal{G}_{\mathrm{poly}}$ compatible with the action of $\mathcal{G}_{\mathrm{poly}}$ on $\mathcal{T}_{\mathrm{poly}} \hookrightarrow \mathcal{T}$.

Remark 3.2.3. For $\tau \in \mathcal{T}$ we will write $\mathcal{Q}_\alpha \tau$ for its canonical projection onto $\mathcal{T}_\alpha$, and define $\|\tau\|_\alpha \stackrel{\mathrm{def}}{=} \|\mathcal{Q}_\alpha \tau\|$. We also write $\mathcal{Q}_{<\alpha}$ for the projection onto $\mathcal{T}_{<\alpha}$, etc.

Another object in the theory of regularity structures is a model. Given an abstract expansion, the model converts it into a concrete distribution describing its local behaviour around every point. We modify the original definition of model in [Hai14], in order to be able to describe time-dependent distributions.

Definition 3.2.4. Given a regularity structure $\mathscr{T} = (\mathcal{T}, \mathcal{G})$, an *inhomogeneous model* (Π, Γ, Σ) consists of the following three elements:

- A collection of maps $\Gamma^t : \mathbb{R}^d \times \mathbb{R}^d \to \mathcal{G}$, parametrised by $t \in \mathbb{R}$, such that

$$\Gamma^t_{xx} = 1 \;, \qquad \Gamma^t_{xy}\Gamma^t_{yz} = \Gamma^t_{xz} \;, \tag{3.7}$$

  for any $x, y, z \in \mathbb{R}^d$ and $t \in \mathbb{R}$, and the action of Γ^t_{xy} on polynomials is given as in Remark 3.2.2 with $h = (0, y - x)$.

- A collection of maps $\Sigma_x : \mathbb{R} \times \mathbb{R} \to \mathcal{G}$, parametrized by $x \in \mathbb{R}^d$, such that, for any $x \in \mathbb{R}^d$ and $s, r, t \in \mathbb{R}$, one has

$$\Sigma^{tt}_x = 1 \;, \qquad \Sigma^{sr}_x \Sigma^{rt}_x = \Sigma^{st}_x \;, \qquad \Sigma^{st}_x \Gamma^t_{xy} = \Gamma^s_{xy} \Sigma^{st}_y \;, \tag{3.8}$$

  and the action of Σ^{st}_x on polynomials is given as in Remark 3.2.2 with $h = (t - s, 0)$.

- A collection of linear maps $\Pi^t_x : \mathcal{T} \to \mathcal{S}'(\mathbb{R}^d)$, such that

$$\Pi^t_y = \Pi^t_x \Gamma^t_{xy} \;, \quad \big(\Pi^t_x X^{(0,\bar{k})}\big)(y) = (y - x)^{\bar{k}} \;, \quad \big(\Pi^t_x X^{(k_0,\bar{k})}\big)(y) = 0 \;, \tag{3.9}$$

  for all $x, y \in \mathbb{R}^d$, $t \in \mathbb{R}$, $\bar{k} \in \mathbb{N}^d$, $k_0 \in \mathbb{N}$ such that $k_0 > 0$.

Moreover, for any $\gamma > 0$ and every $T > 0$, there is a constant C for which the analytic bounds

$$\big|\langle \Pi_x^t \tau, \varphi_x^\lambda \rangle\big| \leq C \|\tau\| \lambda^l \;, \qquad \|\Gamma_{xy}^t \tau\|_m \leq C \|\tau\| |x - y|^{l-m} \;, \tag{3.10a}$$

$$\|\Sigma_x^{st} \tau\|_m \leq C \|\tau\| |t - s|^{(l-m)/\mathfrak{s}_0} \;, \tag{3.10b}$$

hold uniformly over all $\tau \in \mathcal{T}_l$, with $l \in \mathcal{A}$ and $l < \gamma$, all $m \in \mathcal{A}$ such that $m < l$, all $\lambda \in (0, 1]$, all $\varphi \in \mathcal{B}_0^r(\mathbb{R}^d)$ with $r > -\lfloor \min \mathcal{A} \rfloor$, and all $t, s \in [-T, T]$ and $x, y \in \mathbb{R}^d$ such that $|t - s| \leq 1$ and $|x - y| \leq 1$.

In addition, we say that the map Π has time regularity $\delta > 0$, if the bound

$$\big|\langle (\Pi_x^t - \Pi_x^s) \tau, \varphi_x^\lambda \rangle\big| \leq C \|\tau\| |t - s|^{\delta/\mathfrak{s}_0} \lambda^{l - \delta} \;, \tag{3.11}$$

holds for all $\tau \in \mathcal{T}_l$ and the other parameters as before.

Remark 3.2.5. For a model $Z = (\Pi, \Gamma, \Sigma)$, we denote by $\|\Pi\|_{\gamma;T}$, $\|\Gamma\|_{\gamma;T}$ and $\|\Sigma\|_{\gamma;T}$ the smallest constants C such that the bounds on Π, Γ and Σ in (3.10a) and (3.10b) hold. Furthermore, we define

$$\|\!|Z|\!\|_{\gamma;T} \stackrel{\text{def}}{=} \|\Pi\|_{\gamma;T} + \|\Gamma\|_{\gamma;T} + \|\Sigma\|_{\gamma;T} \;.$$

If $\bar{Z} = (\bar{\Pi}, \bar{\Gamma}, \bar{\Sigma})$ is another model, then we also define the "distance" between two models

$$\|\!|Z; \bar{Z}|\!\|_{\gamma;T} \stackrel{\text{def}}{=} \|\Pi - \bar{\Pi}\|_{\gamma;T} + \|\Gamma - \bar{\Gamma}\|_{\gamma;T} + \|\Sigma - \bar{\Sigma}\|_{\gamma;T} \;. \tag{3.12}$$

We note that the norms on the right-hand side still make sense with Γ and Σ viewed as linear maps on $\mathcal{T}$. We also set $\|\Pi\|_{\delta,\gamma;T} \stackrel{\text{def}}{=} \|\Pi\|_{\gamma;T} + C$, where C is the smallest constant such that the bound (3.11) holds, and we define

$$\|\!|Z|\!\|_{\delta,\gamma;T} \stackrel{\text{def}}{=} \|\Pi\|_{\delta,\gamma;T} + \|\Gamma\|_{\gamma;T} + \|\Sigma\|_{\gamma;T} \;.$$

Finally, we define the "distance" $\|\!|Z; \bar{Z}|\!\|_{\delta,\gamma;T}$ as in (3.12).

Remark 3.2.6. In [Hai14, Def. 2.17] the analytic bounds on a model were assumed to hold locally uniformly. In the problems which we aim to consider, the models are periodic in space, which allows us to require the bounds to hold globally.

Remark 3.2.7. For a given model (Π, Γ, Σ) we can define the following two objects

$$\big(\tilde\Pi_{(t,x)}\tau\big)(s,y) = \big(\Pi_x^s \Sigma_x^{st}\tau\big)(y)\;, \qquad \tilde\Gamma_{(t,x),(s,y)} = \Gamma_{xy}^t \Sigma_y^{ts} = \Sigma_x^{ts}\Gamma_{xy}^s\;, \tag{3.13}$$

for $\tau \in \mathcal{T}$. Of course, in general we cannot fix the spatial point y in the definition of $\tilde\Pi$, and we should really write $\big(\big(\tilde\Pi_{(t,x)}\tau\big)(s,\cdot)\big)(\varphi) = \big(\Pi_x^s\Sigma_x^{st}\tau\big)(\varphi)$ instead, for any test function φ, but the notation (3.13) is more suggestive. One can then easily verify that the pair $(\tilde\Pi, \tilde\Gamma)$ is a model in the original sense of [Hai14, Def. 2.17].

3.2.2 Inhomogeneous modelled distributions

Modelled distributions represent abstract expansions in the basis of a regularity structure. In order to be able to describe the singularity coming from the behaviour of our solutions near time 0, we introduce inhomogeneous modelled distributions which admit a certain blow-up as time goes to zero.

Given a regularity structure $\mathscr{T} = (\mathcal{T}, \mathcal{G})$ with a model $Z = (\Pi, \Gamma, \Sigma)$, values $\gamma, \eta \in \mathbb{R}$ and a final time $T > 0$, we consider maps $H : (0,T] \times \mathbb{R}^d \to \mathcal{T}_{<\gamma}$ and define

$$\begin{aligned} \|H\|_{\gamma,\eta;T} \stackrel{\text{def}}{=} &\sup_{t\in(0,T]} \sup_{x\in\mathbb{R}^d} \sup_{l<\gamma} |t|_0^{(l-\eta)\vee 0} \|H_t(x)\|_l \\ &+ \sup_{t\in(0,T]} \sup_{\substack{x\neq y\in\mathbb{R}^d \\ |x-y|\leq 1}} \sup_{l<\gamma} \frac{\|H_t(x) - \Gamma_{xy}^t H_t(y)\|_l}{|t|_0^{\eta-\gamma}|x-y|^{\gamma-l}}\;, \end{aligned} \tag{3.14}$$

where $l \in \mathcal{A}$ in the third supremum. Then the space $\mathcal{D}_T^{\gamma,\eta}$ consists of all such functions H, for which one has

$$|\!|\!| H |\!|\!|_{\gamma,\eta;T} \stackrel{\text{def}}{=} \|H\|_{\gamma,\eta;T} + \sup_{\substack{s\neq t\in(0,T] \\ |t-s|\leq|t,s|_0^{\mathfrak{s}_0}}} \sup_{x\in\mathbb{R}^d} \sup_{l<\gamma} \frac{\|H_t(x) - \Sigma_x^{ts}H_s(x)\|_l}{|t,s|_0^{\eta-\gamma}|t-s|^{(\gamma-l)/\mathfrak{s}_0}} < \infty\;. \tag{3.15}$$

The quantities $|t|_0$ and $|t,s|_0$ used in these definitions were introduced in (3.5). Elements of these spaces will be called *inhomogeneous modelled distributions*.

Remark 3.2.8. The norm in (3.15) depends on Γ and Σ, but does *not* depend on Π; this fact will be crucial in the sequel. When we want to stress the dependency on the model, we will also write $\mathcal{D}_T^{\gamma,\eta}(Z)$.

Remark 3.2.9. In contrast to the singular modelled distributions from [Hai14, Def. 6.2], we do not require the restriction $|x - y| \leq |t, s|_0$ in the second term in (3.14). This is due to the fact that we consider the space and time variables separately, see the proof of Theorem 3.2.21, where this fact is used.

Remark 3.2.10. Since our spaces $\mathcal{D}_T^{\gamma,\eta}$ are almost identical to those of [Hai14, Def. 6.2], the multiplication and differentiation results from [Hai14, Sec. 6] hold also for our definition.

To be able to compare two modelled distributions $H \in \mathcal{D}_T^{\gamma,\eta}(Z)$ and $\bar{H} \in \mathcal{D}_T^{\gamma,\eta}(\bar{Z})$, we define the quantities

$$\begin{aligned}
\|H; \bar{H}\|_{\gamma,\eta;T} &\stackrel{\text{def}}{=} \sup_{t \in (0,T]} \sup_{x \in \mathbb{R}^d} \sup_{l < \gamma} |t|_0^{(l-\eta)\vee 0} \|H_t(x) - \bar{H}_t(x)\|_l \\
&\quad + \sup_{t \in (0,T]} \sup_{\substack{x \neq y \in \mathbb{R}^d \\ |x-y| \leq 1}} \sup_{l < \gamma} \frac{\|H_t(x) - \Gamma^t_{xy} H_t(y) - \bar{H}_t(x) + \bar{\Gamma}^t_{xy} \bar{H}_t(y)\|_l}{|t|_0^{\eta-\gamma} |x-y|^{\gamma-l}} , \\
\vert\!\vert\!\vert H; \bar{H} \vert\!\vert\!\vert_{\gamma,\eta;T} &\stackrel{\text{def}}{=} \|H; \bar{H}\|_{\gamma,\eta;T} \\
&\quad + \sup_{\substack{s \neq t \in (0,T] \\ |t-s| \leq |t,s|_0^{\mathfrak{s}_0}}} \sup_{x \in \mathbb{R}^d} \sup_{l < \gamma} \frac{\|H_t(x) - \Sigma^{ts}_x H_s(x) - \bar{H}_t(x) + \bar{\Sigma}^{ts}_x \bar{H}_s(x)\|_l}{|t,s|_0^{\eta-\gamma} |t-s|^{(\gamma-l)/\mathfrak{s}_0}} .
\end{aligned}$$

The "reconstruction theorem" is one of the key results of the theory of regularity structures. Here is its statement in our current framework.

Theorem 3.2.11. *Let $\mathscr{T} = (\mathcal{T}, \mathcal{G})$ be a regularity structure with $\alpha \stackrel{\text{def}}{=} \min \mathcal{A} < 0$ and let $Z = (\Pi, \Gamma, \Sigma)$ be a model. Then, for every $\eta \in \mathbb{R}$, $\gamma > 0$ and $T > 0$, there is a unique family of linear operators $\mathcal{R}_t : \mathcal{D}_T^{\gamma,\eta}(Z) \to \mathcal{C}^\alpha(\mathbb{R}^d)$, parametrised by $t \in (0, T]$, such that the bound*

$$\left|\langle \mathcal{R}_t H_t - \Pi^t_x H_t(x), \varphi^\lambda_x \rangle\right| \lesssim \lambda^\gamma |t|_0^{\eta-\gamma} \vert\!\vert\!\vert H \vert\!\vert\!\vert_{\gamma,\eta;T} \|\Pi\|_{\gamma;T} , \tag{3.17}$$

holds uniformly in $H \in \mathcal{D}_T^{\gamma,\eta}(Z)$, $t \in (0, T]$, $x \in \mathbb{R}^d$, $\lambda \in (0, 1]$ and $\varphi \in \mathcal{B}_0^r(\mathbb{R}^d)$ with $r > -\lfloor \alpha \rfloor$.

If furthermore the map Π has time regularity $\delta > 0$, then, for any $\tilde{\delta} \in (0, \delta]$ such that $\tilde{\delta} \leq (m - \zeta)$ for all $\zeta, m \in ((-\infty, \gamma) \cap \mathcal{A}) \cup \{\gamma\}$ such that $\zeta < m$, the *function $t \mapsto \mathcal{R}_t H_t$ satisfies*

$$\|\mathcal{R}H\|_{\mathcal{C}^{\tilde{\delta},\alpha}_{\eta-\gamma,T}} \lesssim \|\Pi\|_{\delta,\gamma;T}\big(1 + \|\Sigma\|_{\gamma;T}\big) \|\!|H|\!\|_{\gamma,\eta;T} \;. \tag{3.18}$$

Let $\bar{Z} = (\bar{\Pi}, \bar{\Gamma}, \bar{\Sigma})$ be another model for the same regularity structure, and let $\bar{\mathcal{R}}_t$ be the operator as above, but for the model $\bar{Z}$. Moreover, let the maps Π and $\bar{\Pi}$ have time regularities $\delta > 0$. Then, for every $H \in \mathcal{D}^{\gamma,\eta}_T(Z)$ and $\bar{H} \in \mathcal{D}^{\gamma,\eta}_T(\bar{Z})$, the maps $t \mapsto \mathcal{R}_t H_t$ and $t \mapsto \bar{\mathcal{R}}_t \bar{H}_t$ satisfy

$$\|\mathcal{R}H - \bar{\mathcal{R}}\bar{H}\|_{\mathcal{C}^{\tilde{\delta},\alpha}_{\eta-\gamma,T}} \lesssim \|\!|H; \bar{H}|\!\|_{\gamma,\eta;T} + \|\!|Z; \bar{Z}|\!\|_{\delta,\gamma;T} \;, \tag{3.19}$$

for any $\tilde{\delta}$ as above, and where the proportionality constant depends on $\|\!|H|\!\|_{\gamma,\eta;T}$, $\|\!|\bar{H}|\!\|_{\gamma,\eta;T}$, $\|\!|Z|\!\|_{\delta,\gamma;T}$ and $\|\!|\bar{Z}|\!\|_{\delta,\gamma;T}$.

Proof. Existence and uniqueness of the maps $\mathcal{R}_t$, as well as the bound (3.17), follow from [Hai14, Thm. 3.10]. The uniformity in time in (3.17) follows from the uniformity of the corresponding bounds in [Hai14, Thm. 3.10].

To prove that $t \mapsto \mathcal{R}_t H_t$ belongs to $\mathcal{C}^{\tilde{\delta},\alpha}_{\eta-\gamma}\big([0,T], \mathbb{R}^d\big)$, we will first bound $\langle \mathcal{R}_t H_t, \varrho^\lambda_x \rangle$, for $\lambda \in (0,1]$, $x \in \mathbb{R}^d$ and $\varrho \in \mathcal{B}^r_0(\mathbb{R}^d)$. Using (3.17) and properties of Π and H we get

$$\begin{aligned} |\langle \mathcal{R}_t H_t, \varrho^\lambda_x \rangle| &\leq |\langle \mathcal{R}_t H_t - \Pi^t_x H_t(x), \varrho^\lambda_x \rangle| + |\langle \Pi^t_x H_t(x), \varrho^\lambda_x \rangle| \\ &\lesssim \lambda^\gamma |t|_0^{\eta-\gamma} + \sum_{\zeta \in [\alpha,\gamma) \cap \mathcal{A}} \lambda^\zeta |t|_0^{(\eta-\zeta)\wedge 0} \lesssim \lambda^\alpha |t|_0^{\eta-\gamma} \;, \end{aligned} \tag{3.20}$$

where the proportionality constant is affine in $\|H\|_{\gamma,\eta;T} \|\Pi\|_{\gamma;T}$, and α is the minimal homogeneity in $\mathcal{A}$.

In order to obtain the time regularity of $t \mapsto \mathcal{R}_t H_t$, we show that the distribution $\zeta^{st}_x \stackrel{\text{def}}{=} \Pi^t_x H_t(x) - \Pi^s_x H_s(x)$ satisfies the bound

$$|\langle \zeta^{st}_x - \zeta^{st}_y, \varrho^\lambda_x \rangle| \lesssim |t-s|^{\tilde{\delta}/\mathfrak{s}_0} |s,t|_0^{\eta-\gamma} |x-y|^{\gamma-\tilde{\delta}-\alpha} \lambda^\alpha \;, \tag{3.21}$$

uniformly over all $x, y \in \mathbb{R}^d$ such that $\lambda \leq |x-y| \leq 1$, all $s,t \in \mathbb{R}$, and for any value of $\tilde{\delta}$ as in the statement of the theorem. To this end, we consider two regimes: $|x-y| \leq |t-s|^{1/\mathfrak{s}_0}$ and $|x-y| > |t-s|^{1/\mathfrak{s}_0}$.

In the first case, when $|x-y| \leq |t-s|^{1/\mathfrak{s}_0}$, we write, using Definition 3.2.4,

$$\zeta_x^{st} - \zeta_y^{st} = \Pi_x^t \big(H_t(x) - \Gamma_{xy}^t H_t(y)\big) - \Pi_x^s \big(H_s(x) - \Gamma_{xy}^s H_s(y)\big), \tag{3.22}$$

and bound these two terms separately. From the properties (3.10a) and (3.15) we get

$$\begin{aligned} |\langle \Pi_x^t \big(H_t(x) - \Gamma_{xy}^t H_t(y)\big), \varrho_x^\lambda \rangle| &\lesssim \sum_{\zeta \in [\alpha,\gamma) \cap \mathcal{A}} \lambda^\zeta \|H_t(x) - \Gamma_{xy}^t H_t(y)\|_\zeta \\ &\lesssim \sum_{\zeta \in [\alpha,\gamma) \cap \mathcal{A}} \lambda^\zeta |x-y|^{\gamma-\zeta} |t|_0^{\eta-\gamma} \lesssim \lambda^\alpha |x-y|^{\gamma-\alpha} |t|_0^{\eta-\gamma} , \end{aligned} \tag{3.23}$$

where we have exploited the condition $|x-y| \geq \lambda$. Recalling now the case we consider, we can bound the last expression by the right-hand side of (3.21). The same estimate holds for the second term in (3.22).

Now, we will consider the case $|x-y| > |t-s|^{1/\mathfrak{s}_0}$. In this regime we use the definition of model and write

$$\begin{aligned} \zeta_x^{st} - \zeta_y^{st} = \big(\Pi_x^t - \Pi_x^s\big)\big(H_t(x) - \Gamma_{xy}^t H_t(y)\big) + \Pi_x^s \big(1 - \Sigma_x^{st}\big)\big(H_t(x) - \Gamma_{xy}^t H_t(y)\big) \\ - \Pi_x^s \big(H_s(x) - \Sigma_x^{st} H_t(x)\big) + \Pi_y^s \big(H_s(y) - \Sigma_y^{st} H_t(y)\big) . \end{aligned} \tag{3.24}$$

The first term can be bounded exactly as (3.23), but using this time (3.11), i.e.

$$|\langle \big(\Pi_x^t - \Pi_x^s\big)\big(H_t(x) - \Gamma_{xy}^t H_t(y)\big), \varrho_x^\lambda \rangle| \lesssim \lambda^{\alpha-\delta} |x-y|^{\gamma-\alpha} |t|_0^{\eta-\gamma} |t-s|^{\delta/\mathfrak{s}_0} .$$

In order to estimate the second term in (3.24), we first notice that from (3.10b) and (3.15) we get

$$\begin{aligned} \|\big(1 - \Sigma_x^{st}\big)\big(H_t(x) - \Gamma_{xy}^t H_t(y)\big)\|_\zeta &\lesssim \sum_{\zeta < m < \gamma} |t-s|^{(m-\zeta)/\mathfrak{s}_0} \|H_t(x) - \Gamma_{xy}^t H_t(y)\|_m \\ &\lesssim \sum_{\zeta < m < \gamma} |t-s|^{(m-\zeta)/\mathfrak{s}_0} |x-y|^{\gamma-m} |t|_0^{\eta-\gamma} \lesssim |t-s|^{\tilde{\delta}/\mathfrak{s}_0} |x-y|^{\gamma-\tilde{\delta}-\zeta} |t|_0^{\eta-\gamma}, \end{aligned} \tag{3.25}$$

for any $\tilde{\delta} \leq \min_{m > \zeta \in \mathcal{A}} (m - \zeta)$, where we have used the assumption on the time variables. Hence, for the second term in (3.24) we have

$$|\langle \Pi_x^s \big(1 - \Sigma_x^{st}\big)\big(H_t(x) - \Gamma_{xy}^t H_t(y)\big), \varrho_x^\lambda \rangle|$$

$$\lesssim |t-s|^{\tilde{\delta}/\mathfrak{s}_0}|t|_0^{\eta-\gamma}\sum_{\zeta<\gamma}\lambda^\zeta|x-y|^{\gamma-\tilde{\delta}-\zeta}\;.$$

Since $|x-y|\geq\lambda$ and $\zeta\geq\alpha$, the estimate (3.21) holds for this expression.

The third term in (3.24) we bound using the properties (3.10a) and (3.15) by

$$\begin{aligned}|\langle\Pi_x^s(H_s(x)-\Sigma_x^{st}H_t(x)),\varrho_x^\lambda\rangle| &\lesssim \sum_{\zeta<\gamma}\lambda^\zeta\|H_s(x)-\Sigma_x^{st}H_t(x)\|_\zeta \\ &\lesssim \sum_{\zeta<\gamma}\lambda^\zeta|t-s|^{(\gamma-\zeta)/\mathfrak{s}_0}|t,s|_0^{\eta-\gamma}\;.\end{aligned} \tag{3.26}$$

It follows from $|x-y|\geq\lambda$, $|x-y|>|t-s|^{1/\mathfrak{s}_0}$ and $\zeta\geq\alpha$, that the latter can be estimated as in (3.21), when $\tilde{\delta}\leq\min\{\gamma-\zeta:\zeta\in\mathcal{A},\,\zeta<\gamma\}$. The same bound holds for the last term in (3.24), and this finishes the proof of (3.21).

In view of the bound (3.21) and [Hai14, Prop. 3.25], we conclude that

$$|\langle\mathcal{R}_tH_t-\mathcal{R}_sH_s-\zeta_x^{st},\varrho_x^\lambda\rangle|\lesssim|t-s|^{\tilde{\delta}/\mathfrak{s}_0}\lambda^{\gamma-\tilde{\delta}}|s,t|_0^{\eta-\gamma}\;, \tag{3.27}$$

uniformly over $s,t\in\mathbb{R}$ and the other parameters as in (3.17). Thus, we can write

$$\langle\mathcal{R}_tH_t-\mathcal{R}_sH_s,\varrho_x^\lambda\rangle=\langle\mathcal{R}_tH_t-\mathcal{R}_sH_s-\zeta_x^{st},\varrho_x^\lambda\rangle+\langle\zeta_x^{st},\varrho_x^\lambda\rangle\;,$$

where the first term is bounded in (3.27). The second term we can write as

$$\begin{aligned}\langle\zeta_x^{st},\varrho_x^\lambda\rangle=\langle\big(\Pi_x^t-\Pi_x^s\big)H_t(x),\varrho_x^\lambda\rangle&+\langle\Pi_x^s\big(H_t(x)-\Sigma_x^{ts}H_s(x)\big),\varrho_x^\lambda\rangle\\&+\langle\Pi_x^s\big(\Sigma_x^{ts}-1\big)H_s(x),\varrho_x^\lambda\rangle\;,\end{aligned}$$

which can be bounded by $|t-s|^{\tilde{\delta}/\mathfrak{s}_0}\lambda^{\alpha-\tilde{\delta}}|s,t|_0^{\eta-\gamma}$, using (3.11), (3.26) and (3.10b). Here, in order to estimate the last term, we act similarly to (3.25). Combining all these bounds together, we conclude that

$$|\langle\mathcal{R}_tH_t-\mathcal{R}_sH_s,\varrho_x^\lambda\rangle|\lesssim|t-s|^{\tilde{\delta}/\mathfrak{s}_0}\lambda^{\alpha-\tilde{\delta}}|s,t|_0^{\eta-\gamma}\;, \tag{3.28}$$

which finishes the proof of the claim.

The bound (3.19) can be shown in a similar way. More precisely, similarly to (3.20) and using [Hai14, Eq. 3.4], we can show that

$$|\langle \mathcal{R}_t H_t - \bar{\mathcal{R}}_t \bar{H}_t, \varrho_x^\lambda \rangle| \lesssim \lambda^\alpha |t|_0^{\eta-\gamma} \big(\|\Pi\|_{\gamma;T} \|H; \bar{H}\|_{\gamma,\eta;T} + \|\Pi - \bar{\Pi}\|_{\gamma;T} \|\bar{H}\|_{\gamma,\eta;T} \big).$$

Denoting $\bar{\zeta}_x^{st} \stackrel{\text{def}}{=} \bar{\Pi}_x^t \bar{H}_t(x) - \bar{\Pi}_x^s \bar{H}_s(x)$ and acting as above, we can prove an analogue of (3.27):

$$\begin{aligned} |\langle \mathcal{R}_t H_t - \bar{\mathcal{R}}_t \bar{H}_t - \mathcal{R}_s H_s + \bar{\mathcal{R}}_s \bar{H}_s - \zeta_x^{st} + \bar{\zeta}_x^{st}, \varrho_x^\lambda \rangle| \\ \lesssim |t-s|^{\tilde{\delta}/\mathfrak{s}_0} \lambda^{\gamma-\tilde{\delta}} |s,t|_0^{\eta-\gamma} \big(\|H; \bar{H}\|_{\gamma,\eta;T} + \|Z; \bar{Z}\|_{\delta,\gamma;T} \big) , \end{aligned}$$

with the values of $\tilde{\delta}$ as before. Finally, similarly to (3.28) we get

$$\begin{aligned} |\langle \mathcal{R}_t H_t - \bar{\mathcal{R}}_t \bar{H}_t - \mathcal{R}_s H_s + \bar{\mathcal{R}}_s \bar{H}_s, \varrho_x^\lambda \rangle| \lesssim |t-s|^{\tilde{\delta}/\mathfrak{s}_0} \lambda^{\alpha-\tilde{\delta}} |s,t|_0^{\eta-\gamma} \\ \times \big(\|H; \bar{H}\|_{\gamma,\eta;T} + \|Z; \bar{Z}\|_{\delta,\gamma;T} \big) , \end{aligned}$$

which finishes the proof. □

Definition 3.2.12. We will call the map $\mathcal{R}$, introduced in Theorem 3.2.11, the *reconstruction operator*, and we will always postulate in what follows that $\mathcal{R}_t = 0$, for $t \leq 0$.

Remark 3.2.13. One can see that the map $\tilde{\mathcal{R}}(t, \cdot) \stackrel{\text{def}}{=} \mathcal{R}_t(\cdot)$ is the reconstruction operator for the model (3.13) in the sense of [Hai14, Thm. 3.10].

3.2.3 Convolutions with singular kernels

In the definition of a mild solution to a parabolic stochastic PDE, convolutions with singular kernels are involved. In particular Schauder estimates plays a key role. To describe this on the abstract level, we introduce the abstract integration map.

Definition 3.2.14. Given a regularity structure $\mathscr{T} = (\mathcal{T}, \mathcal{G})$, a linear map $\mathcal{I} : \mathcal{T} \to \mathcal{T}$ is said to be an *abstract integration map* of order $\beta > 0$ if it satisfies the following properties:

- One has $\mathcal{I} : \mathcal{T}_m \to \mathcal{T}_{m+\beta}$, for every $m \in \mathcal{A}$ such that $m + \beta \in \mathcal{A}$.
- For every $\tau \in \mathcal{T}_{\text{poly}}$, one has $\mathcal{I}\tau = 0$, where $\mathcal{T}_{\text{poly}} \subset \mathcal{T}$ contains the polynomial part of $\mathcal{T}$ and was introduced in Remark 3.2.2.

- One has $\mathcal{I}\Gamma\tau - \Gamma\mathcal{I}\tau \in \mathcal{T}_{\text{poly}}$, for every $\tau \in \mathcal{T}$ and $\Gamma \in \mathcal{G}$.

Remark 3.2.15. The second and third properties are dictated by the special role played by polynomials in the Taylor expansion. One can find a more detailed motivation for this definition in [Hai14, Sec. 5]. In general, we also allow for the situation where $\mathcal{I}$ has a domain which isn't all of $\mathcal{T}$.

Now, we will define the singular kernels, convolutions with which we are going to describe.

Definition 3.2.16. A function $K : \mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{R}$ is regularising of order $\beta > 0$, if there is a constant $r > 0$ such that we can decompose

$$K = \sum_{n \geq 0} K^{(n)} \; , \tag{3.29}$$

in such a way that each term $K^{(n)}$ is supported in $\{z \in \mathbb{R}^{d+1} : \|z\|_{\mathfrak{s}} \leq c2^{-n}\}$ for some $c > 0$, satisfies

$$|D^k K^{(n)}(z)| \lesssim 2^{(|\mathfrak{s}|-\beta+|k|_{\mathfrak{s}})n} \; , \tag{3.30}$$

for every multiindex k with $|k|_{\mathfrak{s}} \leq r$, and annihilates every polynomial of scaled degree r, i.e. for every $k \in \mathbb{N}^{d+1}$ such that $|k|_{\mathfrak{s}} \leq r$ it satisfies

$$\int_{\mathbb{R}^{d+1}} z^k K^{(n)}(z) \, dz = 0 \; . \tag{3.31}$$

Now, we will describe the action of a model on the abstract integration map. When it is convenient for us, we will write $K_t(x) = K(z)$, for $z = (t, x)$.

Definition 3.2.17. Let $\mathcal{I}$ be an abstract integration map of order β for a regularity structure $\mathscr{T} = (\mathcal{T}, \mathcal{G})$, let $Z = (\Pi, \Gamma, \Sigma)$ be a model and let K be regularising of order β with $r > -\lfloor \min \mathcal{A} \rfloor$. We say that Z realises K for $\mathcal{I}$, if for every $\alpha \in \mathcal{A}$ and every $\tau \in \mathcal{T}_\alpha$ one has the identity

$$\Pi_x^t (\mathcal{I}\tau + \mathcal{J}_{t,x}\tau)(y) = \int_{\mathbb{R}} \langle \Pi_x^s \Sigma_x^{st} \tau, K_{t-s}(y - \cdot) \rangle \, ds \; , \tag{3.32}$$

where the polynomial $\mathcal{J}_{t,x}\tau$ is defined by

$$\mathcal{J}_{t,x}\tau \stackrel{\text{def}}{=} \sum_{|k|_\mathfrak{s}<\alpha+\beta} \frac{X^k}{k!} \int_\mathbb{R} \langle \Pi_x^s \Sigma_x^{st}\tau, D^k K_{t-s}(x-\cdot)\rangle\, ds\;, \tag{3.33}$$

with $k \in \mathbb{N}^{d+1}$ and the derivative D^k in time-space. Moreover, we require that

$$\begin{aligned} \Gamma_{xy}^t\big(\mathcal{I} + \mathcal{J}_{t,y}\big) &= \big(\mathcal{I} + \mathcal{J}_{t,x}\big)\Gamma_{xy}^t\;, \\ \Sigma_x^{st}\big(\mathcal{I} + \mathcal{J}_{t,x}\big) &= \big(\mathcal{I} + \mathcal{J}_{s,x}\big)\Sigma_x^{st}\;, \end{aligned} \tag{3.34}$$

for all $s, t \in \mathbb{R}$ and $x, y \in \mathbb{R}^d$.

Remark 3.2.18. We define the integrals in (3.32) and (3.33) as sums of the same integrals, but using the functions $K^{(n)}$ from the expansion (3.29). Since these integrals coincide with those from [Hai14] for the model (3.13), it follows from [Hai14, Lem. 5.19] that these sums converge absolutely, and hence the expressions in (3.32) and (3.33) are well defined.

Remark 3.2.19. The identities (3.34) should be viewed as defining $\Gamma_{xy}^t\mathcal{I}\tau$ and $\Sigma_x^{st}\mathcal{I}\tau$ in terms of $\Gamma_{xy}^t\tau$, $\Sigma_x^{st}\tau$, and (3.33).

With all these notations at hand we introduce the following operator acting on modelled distribution $H \in \mathcal{D}_T^{\gamma,\eta}(Z)$ with $\gamma + \beta > 0$:

$$\big(\mathcal{K}_\gamma H\big)_t(x) \stackrel{\text{def}}{=} \mathcal{I}H_t(x) + \mathcal{J}_{t,x}H_t(x) + \big(\mathcal{N}_\gamma H\big)_t(x)\;. \tag{3.35}$$

Here, the last term is $\mathcal{T}_{\text{poly}}$-valued and is given by

$$\big(\mathcal{N}_\gamma H\big)_t(x) \stackrel{\text{def}}{=} \sum_{|k|_\mathfrak{s}<\gamma+\beta} \frac{X^k}{k!} \int_\mathbb{R} \langle \mathcal{R}_s H_s - \Pi_x^s \Sigma_x^{st} H_t(x), D^k K_{t-s}(x-\cdot)\rangle\, ds\;, \tag{3.36}$$

where as before $k \in \mathbb{N}^{d+1}$ and the derivative D^k is in time-space, see Definition 3.2.12 for consistency of notation.

Remark 3.2.20. It follows from Remark 3.2.13 and the proof of [Hai14, Thm. 5.12], that the integral in (3.36) is well-defined, if we express it as a sum of the respective integrals with the functions $K^{(n)}$ in place of K. See also the definition of the operator $\mathbf{R}^+$ in [Hai14, Sec. 7.1].

The modelled distribution $\mathcal{K}_\gamma H$ represents the space-time convolution of H with K, and the following result shows that this action "improves" regularity by β.

Theorem 3.2.21. *Let $\mathscr{T} = (\mathcal{T}, \mathcal{G})$ be a regularity structure with the minimal homogeneity α, let $\mathcal{I}$ be an abstract integration map of an integer order $\beta > 0$, let K be a singular function regularising by β, and let $Z = (\Pi, \Gamma, \Sigma)$ be a model, which realises K for $\mathcal{I}$. Furthermore, let $\gamma > 0$, $\eta < \gamma$, $\eta > -\mathfrak{s}_0$, $\gamma < \eta + \mathfrak{s}_0$, $\gamma + \beta \notin \mathbb{N}$, $\alpha + \beta > 0$ and $r > -\lfloor \alpha \rfloor$, $r > \gamma + \beta$ in Definition 3.2.16.*

Then $\mathcal{K}_\gamma$ maps $\mathcal{D}_T^{\gamma,\eta}(Z)$ into $\mathcal{D}_T^{\bar{\gamma},\bar{\eta}}(Z)$, where $\bar{\gamma} = \gamma + \beta$, $\bar{\eta} = \eta \wedge \alpha + \beta$, and for any $H \in \mathcal{D}_T^{\gamma,\eta}(Z)$ the following bound holds

$$\|\!|\mathcal{K}_\gamma H|\!\|_{\bar{\gamma},\bar{\eta};T} \lesssim \|\!|H|\!\|_{\gamma,\eta;T} \|\Pi\|_{\gamma;T} \|\Sigma\|_{\gamma;T} \big(1 + \|\Gamma\|_{\bar{\gamma};T} + \|\Sigma\|_{\bar{\gamma};T}\big) \;. \tag{3.37}$$

Furthermore, for every $t \in (0, T]$, one has the identity

$$\mathcal{R}_t \big(\mathcal{K}_\gamma H\big)_t(x) = \int_0^t \langle \mathcal{R}_s H_s, K_{t-s}(x - \cdot)\rangle \, ds \;. \tag{3.38}$$

Let $\bar{Z} = (\bar{\Pi}, \bar{\Gamma}, \bar{\Sigma})$ be another model realising K for $\mathcal{I}$, which satisfies the same assumptions, and let $\bar{\mathcal{K}}_\gamma$ be defined by (3.35) *for this model. Then one has*

$$\|\!|\mathcal{K}_\gamma H; \bar{\mathcal{K}}_\gamma \bar{H}|\!\|_{\bar{\gamma},\bar{\eta};T} \lesssim \|\!|H; \bar{H}|\!\|_{\gamma,\eta;T} + \|\!|Z; \bar{Z}|\!\|_{\bar{\gamma};T} \;, \tag{3.39}$$

for all $H \in \mathcal{D}_T^{\gamma,\eta}(Z)$ and $\bar{H} \in \mathcal{D}_T^{\gamma,\eta}(\bar{Z})$. Here, the proportionality constant depends on $\|\!|H|\!\|_{\gamma,\eta;T}$, $\|\!|\bar{H}|\!\|_{\gamma,\eta;T}$ and the norms on the models Z and $\bar{Z}$ involved in the estimate (3.37).

Proof. In view of Remarks 3.2.7 and 3.2.13, the required bounds on the components of $(\mathcal{K}_\gamma H)_t(x)$ and $(\mathcal{K}_\gamma H)_t(x) - \Sigma_x^{ts}(\mathcal{K}_\gamma H)_s(x)$, as well as on the components with non-integer homogeneities of $(\mathcal{K}_\gamma H)_t(y) - \Gamma_{yx}^t(\mathcal{K}_\gamma H)_t(x)$, can be obtained in exactly the same way as in [Hai14, Prop. 6.16]. See the definition of the operator $\mathbf{R}^+$ in [Hai14, Sec. 7.1].

In order to get the required bounds on the elements of the modelled distribution $(\mathcal{K}_\gamma H)_t(x) - \Gamma_{xy}^t(\mathcal{K}_\gamma H)_t(y)$ with integer homogeneities, we need to modify the proof of [Hai14, Prop. 6.16]. The problem is that our definition of modelled distributions is slightly different than the one in [Hai14, Def. 6.2], see Remark 3.2.9. That's why we have to consider only two regimes, $c2^{-n+1} \leq |x - y|$ and $c2^{-n+1} > |x - y|$, in the proof of [Hai14, Prop. 6.16], where c is from Definition 3.2.16. The only place in the proof, which requires a special treatment, is the derivation of the estimate

$$\Big| \int_{\mathbb{R}} \langle \mathcal{R}_s H_s - \Pi_x^s H_s(x), D^k K_{t-s}^{(n)}(x-\cdot) \rangle \, ds \Big| \lesssim 2^{(|k|_\mathfrak{s} - \gamma - \beta)n} |t|_0^{\eta-\gamma} ,$$

which in our case follows trivially from Theorem 3.2.11 and Definition 3.2.16. Here is the place where we need $\gamma - \eta < \mathfrak{s}_0$, in order to have an integrable singularity. Here, we use the same argument as in the proof of [Hai14, Thm. 7.1] to make sure that the time interval does not increase.

With respective modifications of the proof of [Hai14, Prop. 6.16] we can also show that (3.38) and (3.39) hold. □

3.3 Solutions to parabolic stochastic PDEs

We consider a general parabolic stochastic PDE of the form

$$\partial_t u = Au + F(u, \xi) , \qquad u(0, \cdot) = u_0(\cdot) , \tag{3.40}$$

on $\mathbb{R}_+ \times \mathbb{R}^d$, where u_0 is the initial data, ξ is a rough noise, F is a function in u and ξ, which depends in general on the space-time point z and which is affine in ξ, and A is a differential operator such that $\partial_t - A$ has a Green's function G, i.e. G is the distributional solution of $(\partial_t - A)G = \delta_0$. Then we require the following assumption to be satisfied.

Assumption 3.3.1. *The operator A is given by $Q(\nabla)$, for Q a homogeneous polynomial on $\mathbb{R}^d$ of some even degree $\beta > 0$. Its Green's function $G : \mathbb{R}^{d+1} \setminus \{0\} \mapsto \mathbb{R}$ is smooth, non-anticipative, i.e. $G_t = 0$ for $t \leq 0$, and for $\lambda > 0$ satisfies the scaling relation*

$$\lambda^d G_{\lambda^\beta t}(\lambda x) = G_t(x) .$$

Remark 3.3.2. One can find in [Hör55] precise conditions on Q such that G satisfies Assumption 3.3.1.

In order to apply the abstract integration developed in the previous section, we would like the localised singular part of G to have the properties from Definition 3.2.16. The following result, following from [Hai14, Lem. 7.7], shows that we can do this.

Lemma 3.3.3. *Let us consider functions u supported in $\mathbb{R}_+ \times \mathbb{R}^d$ and periodic in the spatial variable with some fixed period. If Assumption 3.3.1 is satisfied with some $\beta > 0$, then we can write $G = K + R$, in such a way that the identity*

$$\big(G \star u\big)(z) = \big(K \star u\big)(z) + \big(R \star u\big)(z)\;,$$

holds for every such function u and every $z \in (-\infty, 1] \times \mathbb{R}^d$, where $\star$ is the space-time convolution. Furthermore, K has the properties from Definition 3.2.16 with the parameters β and some arbitrary (but fixed) value r, and the scaling $\mathfrak{s} = (\beta, 1, \ldots, 1)$. The function R is smooth, non-anticipative and compactly supported.

In particular, it follows from Lemma 3.3.3 that for any $\gamma > 0$ and any periodic $\zeta_t \in \mathcal{C}^\alpha\big(\mathbb{R}^d\big)$, with $t \in \mathbb{R}$ and with probably an integrable singularity at $t = 0$, we can define

$$\left(R_\gamma \zeta\right)_t(x) \stackrel{\text{def}}{=} \sum_{|k|_\mathfrak{s} < \gamma} \frac{X^k}{k!} \int_{\mathbb{R}} \langle \zeta_s, D^k R_{t-s}(x - \cdot)\rangle \, ds\;, \tag{3.41}$$

where $k \in \mathbb{N}^{d+1}$ and D^k is taken in time-space.

3.3.1 Regularity structures for locally subcritical stochastic PDEs

In this section we provide conditions on the equation (3.40), under which one can build a regularity structure for it. More precisely, we consider the mild form of equation (3.40):

$$u = G \star F(u, \xi) + S u_0\;, \tag{3.42}$$

where $\star$ is the space-time convolution, S is the semigroup generated by A and G is its fundamental solution. The scope of the equations we can work with is restricted by the *local subcriticality* condition. In order to state this condition, we assume that $\xi \in \mathcal{C}_\mathfrak{s}^\alpha\big(\mathbb{R}^{d+1}\big)$, for some $\alpha < 0$, and we recall that the nonlinearity F in (3.40) was assumed to be affine in ξ and we formally rewrite it in the following way:

- in the formal expression of F we replace ξ by the dummy variable Ξ,
- if $\alpha + \beta \leq 0$, then we replace every occurrence of u in F by the dummy variable U.

Then we make the following assumption on the equation (3.42).

Assumption 3.3.4. *We assume that the obtained formal expression of F is a polynomial in the dummy variables Ξ and U. Furthermore, we associate to each monomial a homogeneity as follows: Ξ has homogeneity α, U has homogeneity $\alpha + \beta$, and the homogeneity of a product is the sum of the homogeneities of the factors. Then we assume that there is no monomial containing both Ξ and U and the monomials different from Ξ have homogeneities strictly greater than α.*

It was shown in [Hai14, Sec. 8.1] that it is possible to build a regularity structure $\mathscr{T} = (\mathcal{T}, \mathcal{G})$ for a locally subcritical equation and to reformulate it as a fixed point problem in an associated space of modelled distributions. We do not want to give a precise description of this regularity structure, see for example [Hai14, Hai15] for details in the case of Φ^4_3. Let us just mention that we can recursively build two sets of symbols, $\mathcal{F}$ and $\mathcal{U}$. The set $\mathcal{F}$ contains Ξ, $\mathbf{1}$, X_i, as well as some of the symbols that can be built recursively from these basic building blocks by the operations

$$\tau \mapsto \mathcal{I}(\tau) , \qquad (\tau, \bar{\tau}) \mapsto \tau\bar{\tau} , \tag{3.43}$$

subject to the equivalences $\tau\bar{\tau} = \bar{\tau}\tau$, $\mathbf{1}\tau = \tau$, and $\mathcal{I}(X^k) = 0$. These symbols are involved in the description of the right hand side of (3.40). The set $\mathcal{U} \subset \mathcal{F}$ on the other hand contains only those symbols which are used in the description of the solution itself, which are either of the form X^k or of the form $\mathcal{I}(\tau)$ with $\tau \in \mathcal{F}$. The model space $\mathcal{T}$ is then defined as $\mathrm{span}\{\tau \in \mathcal{F} \,:\, |\tau| \leq r\}$ for a sufficiently large $r > 0$, the set of all (real) linear combinations of symbols in $\mathcal{F}$ of homogeneity $|\tau| \leq r$, where $\tau \mapsto |\tau|$ is given by

$$|\mathbf{1}| = 0 , \quad |X_i| = \mathfrak{s}_i , \quad |\Xi| = \alpha, \quad |\mathcal{I}(\tau)| = |\tau| + \beta , \quad |\tau\bar{\tau}| = |\tau| + |\bar{\tau}| . \tag{3.44}$$

In the situation of interest, namely the Φ^4_3 model, one chooses $\beta = 2$ and $\alpha = -\frac{5}{2} - \kappa$ for some $\kappa > 0$ sufficiently small. Subcriticality then guarantees that $\mathcal{T}$ is finite-dimensional. We will also write $\mathcal{T}_{\mathcal{U}}$ for the linear span of $\mathcal{U}$ in $\mathcal{T}$.

One can also build a structure group $\mathcal{G}$ acting on $\mathcal{T}$ in such a way that the operation $\mathcal{I}$ satisfies the assumptions of Definition 3.2.14 (corresponding to the convolution operation with the kernel K), and such that it acts on $\mathcal{T}_{\mathrm{poly}}$ by translations as required.

Let now Z be a model realising K for $\mathcal{I}$, we denote by $\mathcal{R}$, $\mathcal{K}_{\bar{\gamma}}$ and R_γ the reconstruction operator, and the corresponding operators (3.35) and (3.41). We also use the notation $\mathcal{P} \stackrel{\text{def}}{=} \mathcal{K}_{\bar{\gamma}} + R_\gamma \mathcal{R}$ for the operator representing convolution with the heat kernel. With these notations at hand, it was shown in [Hai14] that one can associate to (3.42) the fixed point problem in $\mathcal{D}_T^{\gamma,\eta}(Z)$ given by

$$U = \mathcal{P}F(U) + Su_0 \;, \tag{3.45}$$

for a suitable function (which we call again F) which "represents" the nonlinearity of the stochastic PDE in the sense of [Hai14, Sec. 8] and which is such that $\mathcal{I}F(\tau) \in \mathcal{T}$ for every $\tau \in \mathcal{T}_\mathcal{U}$. In our running example, we would take

$$F(\tau) = -\mathcal{Q}_{\leq 0}(\tau^3) + \Xi \;, \tag{3.46}$$

where $\mathcal{Q}_{\leq 0}$ denotes the canonical projection onto $\mathcal{T}_{\leq 0}$ as before[1]. The problem we encounter is that since we impose that our models are functions of time, there exists no model for which $\Pi_x^t \Xi = \xi$ with ξ a typical realisation of space-time white noise. We would like to replace (3.45) by an equivalent fixed point problem that circumvents this problem, and this is the content of the next two sections.

3.3.2 Truncation of regularity structures

In general, as just discussed, we cannot always define a suitable inhomogeneous model for the regularity structure $\mathscr{T} = (\mathcal{T}, \mathcal{G})$, so we introduce the following truncation procedure, which amounts to simply removing the problematic symbols.

Definition 3.3.5. Consider a set of *generators* $\mathcal{F}^{\text{gen}} \subset \mathcal{F}$ such that $\mathcal{F}_{\text{poly}} \subset \mathcal{F}^{\text{gen}}$ and such that $\mathcal{T}^{\text{gen}} \stackrel{\text{def}}{=} \text{span}\{\tau \in \mathcal{F}^{\text{gen}} : |\tau| \leq r\} \subset \mathcal{T}$ is closed under the action of $\mathcal{G}$. We then define the corresponding *generating regularity structure* $\mathscr{T}^{\text{gen}} = (\mathcal{T}^{\text{gen}}, \mathcal{G})$.

Moreover, we define $\hat{\mathcal{F}}$ as the subset of $\mathcal{F}$ generated by $\mathcal{F}^{\text{gen}}$ via the two operations (3.43), and we assume that $\mathcal{F}^{\text{gen}}$ was chosen in such a way that $\mathcal{U} \subset \hat{\mathcal{F}}$, with $\mathcal{U}$ as in the previous section. Finally, we define the *truncated regularity structure* $\hat{\mathscr{T}} = (\hat{\mathcal{T}}, \mathcal{G})$ with $\hat{\mathcal{T}} \stackrel{\text{def}}{=} \text{span}\{\tau \in \hat{\mathcal{F}} : |\tau| \leq r\} \subset \mathcal{T}$.

Remark 3.3.6. Note that $\hat{\mathscr{T}}$ is indeed a regularity structure since $\hat{\mathcal{T}}$ is automatically closed under $\mathcal{G}$. This can easily be verified by induction using the definition of $\mathcal{G}$ given in [Hai14].

[1] The reason for adding this projection is to guarantee that $\mathcal{I}F$ maps $\mathcal{T}_\mathcal{U}$ into $\mathcal{T}$, since we truncated $\mathcal{T}$ at homogeneity r.

A set $\mathcal{F}^{\mathrm{gen}}$ with these properties always exists, because one can take either $\mathcal{F}^{\mathrm{gen}} = \mathcal{F}$ or $\mathcal{F}^{\mathrm{gen}} = \{\Xi\} \cup \mathcal{F}_{\mathrm{poly}}$. In both of these examples, one simply has $\hat{\mathcal{F}} = \mathcal{F}$, but in the case of (1.9), it turns out to be convenient to make a choice for which this is not the case, see Section 5.2.

3.3.3 A general fixed point map

We now reformulate (3.40), with the operator A such that Assumption 3.3.1 is satisfied, using the regularity structure from the previous section, and show that the corresponding fixed point problem admits local solutions. For an initial condition u_0 in (3.40) with "sufficiently nice" behavior at infinity, we can define the function $S_t u_0 : \mathbb{R}^d \to \mathbb{R}$, which has a singularity at $t = 0$, where as before S_t is the semigroup generated by A. In particular, we have a precise description of its singularity, the proof of which is provided in [Hai14, Lem. 7.5]:

Lemma 3.3.7. *For some $\eta < 0$, let $u_0 \in \mathcal{C}^\eta(\mathbb{R}^d)$ be periodic. Then, for every $\gamma > 0$ and every $T > 0$, the map $(t, x) \mapsto S_t u_0(x)$ can be lifted to $\mathcal{D}_T^{\gamma,\eta}$ via its Taylor expansion. Furthermore, one has the bound*

$$\|\!|S u_0|\!\|_{\gamma,\eta;T} \lesssim \|u_0\|_{\mathcal{C}^\eta} \, . \tag{3.47}$$

Before reformulating (3.40), we make some assumptions on its nonlinear term F. For a regularity structure $\mathscr{T} = (\mathcal{T}, \mathcal{G})$, let $\hat{\mathscr{T}} = (\hat{\mathcal{T}}, \mathcal{G})$ be as in Definition 3.3.5 for a suitable set $\mathcal{F}^{\mathrm{gen}}$. In what follows, we consider models on $\hat{\mathscr{T}}$ and denote by $\mathcal{D}_T^{\gamma,\eta}$ the respective spaces of modelled distributions. We also assume that we are given a function $F \colon \mathcal{T}_{\mathcal{U}} \to \mathcal{T}$ as above (for example (3.46)), and we make the following assumption on F.

For some fixed $\bar{\gamma} > 0$, $\eta \in \mathbb{R}$ we choose, for any model Z on $\hat{\mathscr{T}}$, elements $F_0(Z), I_0(Z) \in \mathcal{D}_T^{\bar{\gamma},\eta}(Z)$ such that, for every z, $I_0(z) \in \hat{\mathcal{T}}$, $I_0(z) - \mathcal{I} F_0(z) \in \mathcal{T}_{\mathrm{poly}}$ and such that, setting

$$\hat{F}(z, \tau) \stackrel{\text{def}}{=} F(z, \tau) - F_0(z) \, , \tag{3.48}$$

$\hat{F}(z, \cdot)$ maps $\{I_0(z) + \tau : \tau \in \hat{\mathcal{T}} \cap \mathcal{T}_{\mathcal{U}}\}$ into $\hat{\mathcal{T}}$. Here we suppressed the argument Z for conciseness by writing for example $I_0(z)$ instead of $I_0(Z)(z)$.

Remark 3.3.8. Since it is the *same* structure group $\mathcal{G}$ acting on both $\mathcal{T}$ and $\hat{\mathcal{T}}$, the condition $F_0 \in \mathcal{D}_T^{\bar{\gamma},\eta}$ makes sense for a given model on $\hat{\mathscr{T}}$, even though $F_0(z)$ takes values in all of $\mathcal{T}$ rather than just $\hat{\mathcal{T}}$.

Given such a choice of I_0 and F_0 and given $H : \mathbb{R}^{d+1} \to \hat{\mathcal{T}} \cap \mathcal{T}_{\mathcal{U}}$, we denote by $\hat{F}(H)$ the function

$$\big(\hat{F}(H)\big)_t(x) \stackrel{\text{def}}{=} \hat{F}\big((t,x), H_t(x)\big)\ . \tag{3.49}$$

With this notation, we replace the problem (3.45) by the problem

$$U = \mathcal{P}\hat{F}(U) + Su_0 + I_0\ . \tag{3.50}$$

This shows that one should really think of I_0 as being given by $I_0 = \mathcal{P}F_0$ since, at least formally, this would then turn (3.50) into (3.45). The advantage of (3.50) is that it makes sense for any model on $\hat{\mathscr{T}}$ and does not require a model on all of $\mathscr{T}$.

We then assume that $\hat{F}$, I_0 and F_0 satisfy the following conditions.

Assumption 3.3.9. *In the above context, we assume that there exists $\gamma \geq \bar{\gamma}$ such that, for every $B > 0$ there exists a constant $C > 0$ such that the bounds*

$$\begin{gathered} \|\hat{F}(H); \hat{F}(\bar{H})\|_{\bar{\gamma},\bar{\eta};T} \leq C\big(\|H; \bar{H}\|_{\gamma,\eta;T} + \|Z; \bar{Z}\|_{\gamma;T}\big), \\ \|I_0(Z); I_0(\bar{Z})\|_{\bar{\gamma},\bar{\eta};T} \leq C\|Z; \bar{Z}\|_{\gamma;T}\ , \quad \|F_0(Z); F_0(\bar{Z})\|_{\bar{\gamma},\bar{\eta};T} \leq C\|Z; \bar{Z}\|_{\gamma;T}\ , \end{gathered} \tag{3.51}$$

hold for any two models Z, $\bar{Z}$ with $\|Z\|_{\gamma;T} + \|\bar{Z}\|_{\gamma;T} \leq B$, and for $H \in \mathcal{D}_T^{\gamma,\eta}(Z)$, $\bar{H} \in \mathcal{D}_T^{\gamma,\eta}(\bar{Z})$ such that $\|H\|_{\gamma,\eta;T} + \|\bar{H}\|_{\gamma,\eta;T} \leq B$.

Remark 3.3.10. The bounds in Assumption (3.3.9) can usually be easily checked for a polynomial nonlinearity F in (3.42). See Lemma 5.2.1 below for a respective prove in the case when F is give by (3.46).

The following theorem provides the existence and uniqueness results of a local solution to this equation.

Theorem 3.3.11. *In the described context, let $\alpha \stackrel{\text{def}}{=} \min \hat{\mathcal{A}}$, and an abstract integration map $\mathcal{I}$ be of order $\beta > -\alpha$. Furthermore, let the values $\gamma \geq \bar{\gamma} > 0$ and $\eta, \bar{\eta} \in \mathbb{R}$ from Assumption 3.3.9 satisfy $\eta < \bar{\eta} \wedge \alpha + \beta$, $\gamma < \bar{\gamma} + \beta$ and $\bar{\eta} > -\beta$.*

Then, for every model Z as above, and for every periodic $u_0 \in \mathcal{C}^\eta(\mathbb{R}^d)$, there exists a time $T_\star \in (0, +\infty]$ such that, for every $T < T_\star$ the equation (3.50) *admits a* *unique solution $U \in \mathcal{D}_T^{\gamma,\eta}(Z)$. Furthermore, if $T_\star < \infty$, then*

$$\lim_{T \to T_\star} \|\mathcal{R}_T \mathcal{S}_T(u_0, Z)_T\|_{\mathcal{C}^\eta} = \infty ,$$

where $\mathcal{S}_T : (u_0, Z) \mapsto U$ is the solution map. Finally, for every $T < T_\star$, the solution map $\mathcal{S}_T$ is jointly Lipschitz continuous in a neighborhood around (u_0, Z) in the sense that, for any $B > 0$ there is $C > 0$ such that, if $\bar{U} = \mathcal{S}_T(\bar{u}_0, \bar{Z})$ for some initial data $(\bar{u}_0, \bar{Z})$, then one has the bound $\|\!|U; \bar{U}|\!\|_{\gamma,\eta;T} \leq C\delta$, provided $\|u_0 - \bar{u}_0\|_{\mathcal{C}^\eta} + \|\!|Z; \bar{Z}|\!\|_{\gamma;T} \leq \delta$, for any $\delta \in (0, B]$.

Proof. See [Hai14, Thm. 7.8], combined with [Hai14, Prop. 7.11]. Note that since we consider inhomogeneous models, we have no problems in evaluating $\mathcal{R}_t U_t$. □

Definition 3.3.12. In the setting of Theorem 3.3.11, let U be the unique solution to the equation (3.50) on $[0, T_\star)$. Then for $t < T_\star$ we define the solution to (3.40) by

$$u_t(x) \stackrel{\text{def}}{=} \big(\mathcal{R}_t U_t\big)(x) . \tag{3.52}$$

Remark 3.3.13. If the noise ξ in (3.40) is smooth, so that this equation can be solved in the classical sense, one can see that the reconstruction operator satisfies

$$\big(\mathcal{R}_t U_t\big)(x) = \big(\Pi_x^t U_t(x)\big)(x) ,$$

and the solution (3.52) coincides with the classical solution.

Chapter 4

Discretisations of rough stochastic PDEs

4.1 Introduction

The aim of this chapter is to develop a general framework for spatial discretisations of the parabolic stochastic PDEs of the type (3.1). The class of spatial discretisations we work with are of the form

$$\partial_t u^\varepsilon = A^\varepsilon u^\varepsilon + F^\varepsilon(u^\varepsilon, \xi^\varepsilon) , \tag{4.1}$$

with the spatial variable taking values in the dyadic grid with mesh size $\varepsilon > 0$, where A^ε, ξ^ε and F^ε are discrete approximations of A, ξ and F respectively.

In order to consider spatial discretisations of rough equations, we will use regularity structures to obtain uniform bounds (in ε) on solutions to (4.1) by describing the right hand side via a type of generalised "Taylor expansion" in the neighborhood of any space-time point. The problem of obtaining uniform bounds is then split into the problem of on the one hand obtaining uniform bounds on the objects playing the role of Taylor monomials (these require subtle stochastic cancellations, but are given by explicit formulae), and on the other hand obtaining uniform regularity estimates on the "Taylor coefficients" (these are described implicitly as solutions to a fixed point problem but can be controlled by standard Banach fixed point arguments).

In order to treat the discretised equation (4.1), we introduce a discrete analogue to the concept of "model" introduced in Chapter 3 and we show that the corresponding "reconstruction map" satisfies uniform bounds analogous to the ones available in the continuous case. Moreover, we describe convolutions with discrete kernels on the abstract level and prove a discrete analogue of the Schauder estimates.

Structure of the chapter

This chapter is structured in the following way. In Section 4.2 we define discrete models and modelled distributions. Furthermore, we prove discrete analogues of the reconstruction theorem and the Schauder estimates. Section 4.3 is devoted to the analysis of discrete kernels and solutions to discretised stochastic PDEs. Finally, in Section 4.4 we analyse discrete models built from a Gaussian noise.

4.1.1 Notations and conventions

Throughout this chapter we will use the notation of Chapter 3. Furthermore, in this chapter we will also work with discrete functions $\zeta^\varepsilon \in \mathbb{R}^{\Lambda_\varepsilon^d}$ on the dyadic grid $\Lambda_\varepsilon^d \subset \mathbb{R}^d$ with the mesh size $\varepsilon = 2^{-N}$ for $N \in \mathbb{N}$. In order to compare them with their continuous counterparts $\zeta \in \mathcal{C}^\alpha(\mathbb{R}^d)$ with $\alpha \leq 0$, we introduce the following "distance":

$$\|\zeta;\zeta^\varepsilon\|_{\mathcal{C}^\alpha}^{(\varepsilon)} \stackrel{\text{def}}{=} \sup_{\varphi \in \mathcal{B}_0^r} \sup_{x \in \Lambda_\varepsilon^d} \sup_{\lambda \in [\varepsilon, 1]} \lambda^{-\alpha} |\langle \zeta, \varphi_x^\lambda \rangle - \langle \zeta^\varepsilon, \varphi_x^\lambda \rangle_\varepsilon| \;,$$

where $\langle \cdot, \cdot \rangle_\varepsilon$ is the discrete analogue of the duality pairing on the grid, i.e.

$$\langle \zeta^\varepsilon, \varphi_x^\lambda \rangle_\varepsilon \stackrel{\text{def}}{=} \int_{\Lambda_\varepsilon^d} \zeta^\varepsilon(y) \varphi_x^\lambda(y)\, dy \stackrel{\text{def}}{=} \varepsilon^d \sum_{y \in \Lambda_\varepsilon^d} \zeta^\varepsilon(y) \varphi_x^\lambda(y) \;. \tag{4.2}$$

For space-time distributions / functions ζ and ζ^ε, for $\delta > 0$ and $\eta \leq 0$, we define

$$\|\zeta;\zeta^\varepsilon\|_{\mathcal{C}_{\eta,T}^{\delta,\alpha}}^{(\varepsilon)} \stackrel{\text{def}}{=} \sup_{t \in (0,T]} |t|_0^{-\eta} \|\zeta_t;\zeta_t^\varepsilon\|_{\mathcal{C}^\alpha}^{(\varepsilon)} + \sup_{s \neq t \in (0,T]} |s,t|_0^{-\eta} \frac{\|\delta^{s,t}\zeta;\delta^{s,t}\zeta^\varepsilon\|_{\mathcal{C}^{\alpha-\delta}}^{(\varepsilon)}}{\big(|t-s|^{1/\mathfrak{s}_0} \vee \varepsilon\big)^\delta} \;. \tag{4.3}$$

Furthermore, we define the norm $\|\zeta^\varepsilon\|_{\mathcal{C}_{\eta,T}^{\delta,\alpha}}^{(\varepsilon)}$ in the same way as in (3.3) and (3.5), but using the discrete pairing (4.2), the quantities $|t|_\varepsilon \stackrel{\text{def}}{=} |t|_0 \vee \varepsilon$ and $|s,t|_\varepsilon \stackrel{\text{def}}{=} |s|_\varepsilon \wedge |t|_\varepsilon$ instead of $|t|_0$ and $|s,t|_0$ respectively, and $|t-s|^{1/\mathfrak{s}_0} \vee \varepsilon$ instead of $|t-s|^{1/\mathfrak{s}_0}$. Finally, we denote by $\star_\varepsilon$ the convolution on $\mathbb{R} \times \Lambda_\varepsilon^d$.

4.2 Discrete models and modelled distributions

In order to be able to consider discretisations of the equations whose solutions were provided in Chapter 3, we introduce the discrete counterparts of inhomogeneous models and modelled distributions. In this section we use the following notation: for $N \in \mathbb{N}$, we denote by $\varepsilon \stackrel{\text{def}}{=} 2^{-N}$ the mesh size of the grid $\Lambda_\varepsilon^d \stackrel{\text{def}}{=} \big(\varepsilon\mathbb{Z}\big)^d$, and we fix come scaling $\mathfrak{s} = (\mathfrak{s}_0, 1, \ldots, 1)$ of $\mathbb{R}^{d+1}$ with an integer $\mathfrak{s}_0 > 0$.

4.2.1 Definitions and the reconstruction theorem

Now we define discrete analogues of the objects from Sections 3.2.1 and 3.2.2.

Definition 4.2.1. Given a regularity structure $\mathscr{T}$ and $\varepsilon > 0$, a *discrete model* $(\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ consists of the collections of maps

$$\Pi_x^{\varepsilon,t} : \mathcal{T} \to \mathbb{R}^{\Lambda_\varepsilon^d} , \qquad \Gamma^{\varepsilon,t} : \Lambda_\varepsilon^d \times \Lambda_\varepsilon^d \to \mathcal{G} , \qquad \Sigma_x^\varepsilon : \mathbb{R} \times \mathbb{R} \to \mathcal{G} ,$$

parametrised by $t \in \mathbb{R}$ and $x \in \Lambda_\varepsilon^d$, which have all the algebraic properties of their continuous counterparts in Definition 3.2.4, with the spatial variables restricted to the grid. Additionally, we require $\big(\Pi_x^{\varepsilon,t}\tau\big)(x) = 0$, for all $\tau \in \mathcal{T}_l$ with $l > 0$, and all $x \in \Lambda_\varepsilon^d$ and $t \in \mathbb{R}$.

We define the quantities $\|\Pi^\varepsilon\|_{\gamma;T}^{(\varepsilon)}$ and $\|\Gamma^\varepsilon\|_{\gamma;T}^{(\varepsilon)}$ to be the smallest constants C such that the bounds (3.10a) hold uniformly in $x, y \in \Lambda_\varepsilon^d$, $t \in \mathbb{R}$, $\lambda \in [\varepsilon, 1]$ and with the discrete pairing (4.2) in place of the standard one. The quantity $\|\Sigma^\varepsilon\|_{\gamma;T}^{(\varepsilon)}$ is defined as the smallest constant C such that the bounds

$$\|\Sigma_x^{\varepsilon,st}\tau\|_m \leq C\|\tau\|\big(|t-s|^{1/\mathfrak{s}_0} \vee \varepsilon\big)^{l-m} , \tag{4.4}$$

hold uniformly in $x \in \Lambda_\varepsilon^d$ and the other parameters as in (3.10b).

We measure the time regularity of Π^ε as in (3.11), by substituting the continuous objects by their discrete analogues, and by using $|t-s|^{1/\mathfrak{s}_0} \vee \varepsilon$ instead of $|t-s|^{1/\mathfrak{s}_0}$ on the right-hand side. All the other quantities $\|\cdot\|^{(\varepsilon)}$, $|\!|\!|\cdot|\!|\!|^{(\varepsilon)}$, etc. are defined by analogy with Remark 3.2.5.

Remark 4.2.2. The fact that $\big(\Pi_x^{\varepsilon,t}\tau\big)(x) = 0$ if $|\tau| > 0$ does not follow automatically from the discrete analogue of (3.10a) since these are only assumed to hold for test functions at scale $\lambda \geq \varepsilon$. We use this property in the proof of (4.40).

Remark 4.2.3. The weakening of the continuity property of $\Sigma_x^{\varepsilon,st}$ given by (4.4) will be used in the analysis of the "discrete abstract integration" in Section 4.2.2. It allows us to deal with the fact that the discrete heat kernel is discontinuous at $t = 0$, so we simply use uniform bounds on very small time scales. See [HMW14, Lem. 6.7] for a simple explanation in a related context.

For $\gamma, \eta \in \mathbb{R}$ and $T > 0$, for a discrete model $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ on a regularity structure $\mathscr{T} = (\mathcal{T}, \mathcal{G})$, and for a function $H^\varepsilon : (0, T] \times \Lambda_\varepsilon^d \to \mathcal{T}_{<\gamma}$, we define

$$\|H^\varepsilon\|^{(\varepsilon)}_{\gamma,\eta;T} \stackrel{\text{def}}{=} \sup_{t\in(0,T]} \sup_{x\in\Lambda_\varepsilon^d} \sup_{l<\gamma} |t|_\varepsilon^{(l-\eta)\vee 0} \|H_t^\varepsilon(x)\|_l \\ + \sup_{t\in(0,T]} \sup_{\substack{x\neq y\in\Lambda_\varepsilon^d \\ |x-y|\leq 1}} \sup_{l<\gamma} \frac{\|H_t^\varepsilon(x) - \Gamma_{xy}^{\varepsilon,t} H_t^\varepsilon(y)\|_l}{|t|_\varepsilon^{\eta-\gamma}|x-y|^{\gamma-l}}\;, \tag{4.5}$$

where $l \in \mathcal{A}$. Furthermore, we define the norm

$$\|\!|H^\varepsilon|\!\|^{(\varepsilon)}_{\gamma,\eta;T} \stackrel{\text{def}}{=} \|H^\varepsilon\|^{(\varepsilon)}_{\gamma,\eta;T} + \sup_{\substack{s\neq t\in(0,T] \\ |t-s|\leq|t,s|_0^{s_0}}} \sup_{x\in\Lambda_\varepsilon^d} \sup_{l<\gamma} \frac{\|H_t^\varepsilon(x) - \Sigma_x^{\varepsilon,ts} H_s^\varepsilon(x)\|_l}{|t,s|_\varepsilon^{\eta-\gamma}\big(|t-s|^{1/s_0}\vee\varepsilon\big)^{\gamma-l}}\;, \tag{4.6}$$

where the quantities $|t|_\varepsilon$ and $|t,s|_\varepsilon$ are defined below (4.3). We will call such functions H^ε *discrete modelled distributions*.

Remark 4.2.4. It is easy to see that the properties of multiplication of modeled distributions from [Hai14, Sec. 6.2] can be translated mutatis mutandis to the discrete case.

In contrast to the continuous case, a reconstruction operator of discrete modeled distributions can be defined in a simple way.

Definition 4.2.5. Given a discrete model $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ and a discrete modelled distribution H^ε we define the *discrete reconstruction map* $\mathcal{R}^\varepsilon$ by $\mathcal{R}_t^\varepsilon = 0$ for $t \leq 0$, and

$$\big(\mathcal{R}_t^\varepsilon H_t^\varepsilon\big)(x) \stackrel{\text{def}}{=} \big(\Pi_x^{\varepsilon,t} H_t^\varepsilon(x)\big)(x)\;, \qquad (t,x)\in(0,T]\times\Lambda_\varepsilon^d\;. \tag{4.7}$$

Recalling the definition of the norms from (4.3), the following result is a discrete analogue of Theorem 3.2.11.

Theorem 4.2.6. *Let $\mathscr{T}$ be a regularity structure with $\alpha \stackrel{\text{def}}{=} \min\mathcal{A} < 0$ and $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ be a discrete model. Then the bound*

$$|\langle \mathcal{R}_t^\varepsilon H_t^\varepsilon - \Pi_x^{\varepsilon,t} H_t^\varepsilon(x), \varrho_x^\lambda\rangle_\varepsilon| \lesssim \lambda^\gamma |t|_\varepsilon^{\eta-\gamma} \|H^\varepsilon\|^{(\varepsilon)}_{\gamma,\eta;T} \|\Pi^\varepsilon\|^{(\varepsilon)}_{\gamma;T}\;,$$

holds uniformly in ε (see Remark 4.2.7 below) for all discrete modelled distributions H^ε, all $t\in(0,T]$, $x\in\Lambda_\varepsilon^d$, $\varrho\in\mathcal{B}_0^r(\mathbb{R}^d)$ with $r > -\lfloor\alpha\rfloor$, all $\lambda\in[\varepsilon,1]$.

Let furthermore $\bar Z^\varepsilon = (\bar\Pi^\varepsilon, \bar\Gamma^\varepsilon, \bar\Sigma^\varepsilon)$ be another model for $\mathscr{T}$ with the reconstruction operator $\bar{\mathcal{R}}_t^\varepsilon$, and let the maps Π^ε and $\bar\Pi^\varepsilon$ have time regularities $\delta > 0$.

Then, for any two discrete modelled distributions H^ε and $\bar{H}^\varepsilon$, the maps $t \mapsto \mathcal{R}_t^\varepsilon H_t^\varepsilon$ and $t \mapsto \bar{\mathcal{R}}_t^\varepsilon \bar{H}_t^\varepsilon$ satisfy

$$\|\mathcal{R}^\varepsilon H^\varepsilon\|^{(\varepsilon)}_{\mathcal{C}^{\tilde{\delta},\alpha}_{\eta-\gamma,T}} \lesssim \|\Pi^\varepsilon\|^{(\varepsilon)}_{\delta,\gamma;T}\bigl(1 + \|\Sigma^\varepsilon\|^{(\varepsilon)}_{\gamma;T}\bigr)\|\!|H^\varepsilon|\!\|^{(\varepsilon)}_{\gamma,\eta;T} , \tag{4.8a}$$

$$\|\mathcal{R}^\varepsilon H^\varepsilon - \bar{\mathcal{R}}^\varepsilon \bar{H}^\varepsilon\|^{(\varepsilon)}_{\mathcal{C}^{\tilde{\delta},\alpha}_{\eta-\gamma,T}} \lesssim \|\!|H^\varepsilon; \bar{H}^\varepsilon|\!\|^{(\varepsilon)}_{\gamma,\eta;T} + \|\!|Z^\varepsilon; \bar{Z}^\varepsilon|\!\|^{(\varepsilon)}_{\delta,\gamma;T} , \tag{4.8b}$$

for any $\tilde{\delta}$ as in Theorem 3.2.11. Here, the norms of H^ε and $\bar{H}^\varepsilon$ are defined via the models Z^ε and $\bar{Z}^\varepsilon$ respectively, and the proportionality constants depend on ε only via $\|\!|H^\varepsilon|\!\|^{(\varepsilon)}_{\gamma,\eta;T}$, $\|\!|\bar{H}^\varepsilon|\!\|^{(\varepsilon)}_{\gamma,\eta;T}$, $\|\!|Z^\varepsilon|\!\|^{(\varepsilon)}_{\delta,\gamma;T}$ and $\|\!|\bar{Z}^\varepsilon|\!\|^{(\varepsilon)}_{\delta,\gamma;T}$.

Remark 4.2.7. In the statement of Theorem 4.2.6 and the following results we actually consider a sequence of discrete models and modeled distributions parametrised by $\varepsilon = 2^{-N}$ with $N \in \mathbb{N}$. By "uniformity in ε" we then mean that the estimates hold for all values of ε with a proportionality constant independent of ε.

Remark 4.2.8. To compare a discrete model $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ to a continuous model $Z = (\Pi, \Gamma, \Sigma)$, we can define

$$\begin{aligned}\|\Pi; \Pi^\varepsilon\|^{(\varepsilon)}_{\delta,\gamma;T} &\stackrel{\text{def}}{=} \sup_{\varphi,x,\lambda,l,\tau} \sup_{t\in[-T,T]} \lambda^{-l}|\langle \Pi_x^t \tau, \varphi_x^\lambda\rangle - \langle \Pi_x^{\varepsilon,t}\tau, \varphi_x^\lambda\rangle_\varepsilon| \\ &+ \sup_{\varphi,x,\lambda,l,\tau} \sup_{\substack{s\neq t\in[-T,T] \\ |t-s|\leq 1}} \lambda^{-l+\delta} \frac{|\langle \bigl(\Pi_x^t - \Pi_x^s\bigr)\tau, \varphi_x^\lambda\rangle - \langle \bigl(\Pi_x^{\varepsilon,t} - \Pi_x^{\varepsilon,s}\bigr)\tau, \varphi_x^\lambda\rangle_\varepsilon|}{\bigl(|t-s|^{1/\mathfrak{s}_0} \vee \varepsilon\bigr)^\delta} ,\end{aligned}$$

where the supremum is taken over $\varphi \in \mathcal{B}_0^r$, $x \in \Lambda_\varepsilon^d$, $\lambda \in [\varepsilon, 1]$, $l < \gamma$ and $\tau \in \mathcal{T}_l$ with $\|\tau\| = 1$. In order to compare discrete and continuous modelled distributions, we use the quantities as in (3.16), but with the respective modifications as in (4.6).

Then one can show similarly to (3.19) that for $H \in \mathcal{D}_T^{\gamma,\eta}(Z)$ and a discrete modelled distribution H^ε the maps $t \mapsto \mathcal{R}_t H_t$ and $t \mapsto \mathcal{R}_t^\varepsilon H_t^\varepsilon$ satisfy the estimate

$$\|\mathcal{R}H; \mathcal{R}^\varepsilon H^\varepsilon\|^{(\varepsilon)}_{\mathcal{C}^{\tilde{\delta},\alpha}_{\eta-\gamma,T}} \lesssim \|\!|H; H^\varepsilon|\!\|^{(\varepsilon)}_{\gamma,\eta;T} + \|\!|Z; Z^\varepsilon|\!\|^{(\varepsilon)}_{\delta,\gamma;T} + \varepsilon^\theta ,$$

uniformly in ε (see Remark 4.2.7) for $\tilde{\delta} > 0$ and $\theta > 0$ small enough. We will however not make use of this in the present work.

In order to prove Theorem 4.2.6, we need to introduce a multiresolution analysis and its discrete analogue.

4.2.1.1 Elements of multiresolution analysis

In this section we provide only the very basics of the multiresolution analysis, which are used in the sequel. For a more detailed introduction and for the proofs of the provided results we refer to [Dau92] and [Mey92].

One of the remarkable results of [Dau88] is that for every $r > 0$ there exists a compactly supported function $\varphi \in \mathcal{C}^r(\mathbb{R})$ (called *scaling function*) such that

$$\int_{\mathbb{R}} \varphi(x)\, dx = 1 , \qquad \int_{\mathbb{R}} \varphi(x)\varphi(x+k)\, dx = \delta_{0,k}, \quad k \in \mathbb{Z} , \tag{4.9}$$

where $\delta_{\cdot,\cdot}$ is the Kronecker's delta on $\mathbb{Z}$. Furthermore, if for $n \in \mathbb{N}$ we define the grid $\Lambda_n \stackrel{\text{def}}{=} \{2^{-n}k : k \in \mathbb{Z}\}$ and the family of functions

$$\varphi_x^n(\cdot) \stackrel{\text{def}}{=} 2^{n/2}\varphi\big(2^n(\cdot - x)\big) , \qquad x \in \Lambda_n , \tag{4.10}$$

then there is a finite collection of vectors $\mathcal{K} \subset \Lambda_1$ and a collection of structure constants $\{a_k : k \in \mathcal{K}\}$ such that the *refinement equation*

$$\varphi_x^n = \sum_{k\in\mathcal{K}} a_k \varphi_{x+2^{-n}k}^{n+1} \tag{4.11}$$

holds. Note that the multiplier in (4.10) preserves the L^2-norm of the scaled functions rather than their L^1-norm. It follows immediately from (4.9) and (4.11) that one has the identities

$$\sum_{k\in\mathcal{K}} a_k = 2^{d/2} , \qquad \sum_{k\in\mathcal{K}} a_k a_{k+m} = \delta_{0,m} , \quad m \in \mathbb{Z}^d . \tag{4.12}$$

For a fixed scaling function φ, we denote by $V_n \subset L^2(\mathbb{R})$ the subspace spanned by $\{\varphi_x^n : x \in \Lambda_n\}$. Then the relation (4.11) ensures the inclusion $V_n \subset V_{n+1}$ for every n. It turns out that there is a compactly supported function $\psi \in \mathcal{C}^r(\mathbb{R})$ (called *wavelet function*) such that the space $V_n^\perp$, which is the orthogonal complement of V_n in V_{n+1}, is given by

$$V_n^\perp = \operatorname{span}\{\psi_x^n : x \in \Lambda_n\} ,$$

where ψ_x^n is as in (4.11). Moreover, there are constants $\{b_k : k \in \mathcal{K}\}$, such that the *wavelet equation* holds:

$$\psi_x^n = \sum_{k \in \mathcal{K}} b_k \varphi_{x+2^{-n}k}^{n+1} \,. \tag{4.13}$$

One more useful property of the wavelet function is that it has vanishing moments, in the sense that the identity

$$\int_{\mathbb{R}} \psi(x)\, x^m dx = 0 \tag{4.14}$$

holds for all $m \in \mathbb{N}$ such that $m \le r$. The following theorem is an important result of the multiresolution analysis.

Theorem 4.2.9. *In the context just described, for every $n \in \mathbb{N}$, the set*

$$\{\varphi_x^n : x \in \Lambda_n\} \cup \{\psi_x^m : m \ge n,\, x \in \Lambda_m\} \,, \tag{4.15}$$

forms an orthonormal basis of $L^2(\mathbb{R})$.

The *wavelet decomposition* refers to the expansion of a function $f \in L^2(\mathbb{R})$ (also holds for a large class of tempered distributions) in the basis (4.15). In particular, one has

$$\mathcal{P}_n f \stackrel{\text{def}}{=} \sum_{x \in \Lambda_n} \langle f, \varphi_x^n \rangle \varphi_x^n \to f \tag{4.16}$$

in the respective topology, as $n \to \infty$. Here, $\langle \cdot, \cdot \rangle$ is either the L^2-inner product or the dual pairing, depending on whether f is a function or a distribution.

There is a standard generalization of scaling and wavelet functions to $\mathbb{R}^d$, namely for $n \ge 0$ and $x = (x_1, \ldots, x_d) \in \Lambda_n^d$ we define

$$\varphi_x^n(y) \stackrel{\text{def}}{=} \varphi_{x_1}^n(y_1) \cdots \varphi_{x_d}^n(y_d) \,, \qquad y = (y_1, \ldots, y_d) \in \mathbb{R}^d \,.$$

For these scaling functions we also define V_n as the closed subspace in L^2 spanned by $\{\varphi_x^n : x \in \Lambda_n^d\}$. Then there is a finite set $\mathbf{\Psi}$ of functions on $\mathbb{R}^d$ such that the space $V_n^\perp \stackrel{\text{def}}{=} V_{n+1} \setminus V_n$ is a span of $\{\psi_x^n : \psi \in \mathbf{\Psi},\, x \in \Lambda_n^d\}$, where we define the scaled function ψ_x^n by

$$\psi_x^n(y) \stackrel{\text{def}}{=} 2^{nd/2} \psi\big(2^n(y_1 - x_1), \ldots, 2^n(y_d - x_d)\big) \,.$$

All the results mentioned above can be literally translated from $\mathbb{R}$ to $\mathbb{R}^d$, but of course with $\mathcal{K} \subset \Lambda_1^d$ and with different structure constants $\{a_k : k \in \mathcal{K}\}$ and $\{b_k : k \in \mathcal{K}\}$.

4.2.1.2 An analogue of the multiresolution analysis on the grid

In this section we will develop an analogue of the multiresolution analysis which will be useful for working with functions defined on a dyadic grid. Our construction agrees with the standard discrete wavelets on grid points, but also extends off the grid. To this end, we use the notation of Section 4.2.1.1. We recall furthermore that we use $\varepsilon = 2^{-N}$ for a fixed $N \in \mathbb{N}$.

Let us fix a scaling function $\varphi \in \mathcal{C}_0^r(\mathbb{R})$, for some integer $r > 0$, as in Section 4.2.1.1. For integers $0 \le n \le N$ we define the functions

$$\varphi_x^{N,n}(\cdot) \stackrel{\text{def}}{=} 2^{Nd/2} \langle \varphi_\cdot^N, \varphi_x^n \rangle \;, \qquad x \in \Lambda_n^d \;. \tag{4.17}$$

One has that $\varphi_x^{N,n} \in \mathcal{C}^r(\mathbb{R}^d)$, it is supported in a ball of radius $\mathcal{O}(2^{-n})$ centered at x, it has the same scaling properties as φ_x^n, and it satisfies

$$\varphi_x^{N,N}(y) = 2^{Nd/2} \delta_{x,y} \;, \qquad x, y \in \Lambda_N^d \;, \tag{4.18}$$

where $\delta_{\cdot,\cdot}$ is the Kronecker's delta on Λ_N^d. The last property follows from (4.9). Furthermore, it follows from (4.11) that for $n < N$ these functions satisfy the refinement identity

$$\varphi_x^{N,n} = \sum_{k \in \mathcal{K}} a_k \, \varphi_{x+2^{-n}k}^{N,n+1} \;, \tag{4.19}$$

with the same structure constants $\{a_k : k \in \mathcal{K}\}$ as for the functions φ_x^n. One more consequence of (4.9) is

$$2^{-Nd} \sum_{y \in \Lambda_N^d} \varphi_x^{N,n}(y) = 2^{-nd/2} \;,$$

which obviously holds for $n = N$, and for $n < N$ it can be proved by induction, using (4.19) and (4.12).

The functions $\varphi_x^{N,n}$ inherit many of the crucial properties of the functions φ_x^n, which allows us to use them in the multiresolution analysis. In particular, for $n < N$ and $\psi \in \mathbf{\Psi}$ (the set of wavelet functions, introduced in Section 4.2.1.1), we can define the functions

$$\psi_x^{N,n}(\cdot) \stackrel{\text{def}}{=} 2^{Nd/2}\langle \varphi_\cdot^N, \psi_x^n\rangle\;, \qquad x \in \Lambda_n^d\;,$$

whose properties are similar to those of ψ_x^n. For example, $\psi_x^{N,n} \in \mathcal{C}^r(\mathbb{R})$, and it has the same scaling and support properties as ψ_x^n. Furthermore, it follows from (4.13) that for $n < N$ the following identity holds

$$\psi_x^{N,n} = \sum_{k\in\mathcal{K}} b_k \varphi_{x+2^{-n}k}^{N,n+1}\;, \tag{4.20}$$

with the same constants $\{b_k : k \in \mathcal{K}\}$. It is easy to see that the functions just introduced are not L^2-orthogonal, but still, using (4.12), one can show that a result analogous to Theorem 4.2.9 for the inner product $\langle\cdot,\cdot\rangle_\varepsilon$ holds.

4.2.1.3 Proof of the discrete reconstruction theorem

With the help of the discrete analogue of the multiresolution analysis introduced in the previous section we are ready to prove Theorem 4.2.6.

Proof of Theorem 4.2.6. We take a compactly supported scaling function $\varphi \in \mathcal{C}^r(\mathbb{R}^d)$ of regularity $r > -\lfloor\alpha\rfloor$, where α is as in the statement of the theorem, and build the functions $\varphi_x^{N,n}$ as in (4.17). Furthermore, we define the discrete functions $\zeta_x^{\varepsilon,t} \stackrel{\text{def}}{=} \Pi_x^{\varepsilon,t} H_t^\varepsilon(x)$ and $\zeta_{xy}^{\varepsilon,t} \stackrel{\text{def}}{=} \zeta_y^{\varepsilon,t} - \zeta_x^{\varepsilon,t}$. Then from Definition 4.2.1 we obtain

$$\begin{aligned}\big|\langle \zeta_{xy}^{\varepsilon,t}, \varphi_y^{N,n}\rangle_\varepsilon\big| &\lesssim \|\Pi^\varepsilon\|_{\gamma;T}^{(\varepsilon)} \sum_{l\in[\alpha,\gamma)\cap\mathcal{A}} 2^{-nd/2-ln}\|H_t^\varepsilon(y) - \Gamma_{yx}^{\varepsilon,t}H_t^\varepsilon(x)\|_l \\ &\lesssim \|\Pi^\varepsilon\|_{\gamma;T}^{(\varepsilon)}\|H^\varepsilon\|_{\gamma,\eta;T}^{(\varepsilon)}|t|_\varepsilon^{\eta-\gamma} \sum_{l\in[\alpha,\gamma)\cap\mathcal{A}} 2^{-nd/2-ln}|y-x|^{\gamma-l} \\ &\lesssim \|\Pi^\varepsilon\|_{\gamma;T}^{(\varepsilon)}\|H^\varepsilon\|_{\gamma,\eta;T}^{(\varepsilon)}|t|_\varepsilon^{\eta-\gamma}2^{-nd/2-\alpha n}|y-x|^{\gamma-\alpha}\;, \end{aligned} \tag{4.21}$$

which holds as soon as $|x-y| \geq 2^{-n}$. Moreover, we define

$$\mathcal{R}_t^{\varepsilon,n} H_t^\varepsilon \stackrel{\text{def}}{=} \sum_{y\in\Lambda_n^d} \langle \zeta_y^{\varepsilon,t}, \varphi_y^{N,n}\rangle_\varepsilon\, \varphi_y^{N,n}\;.$$

It follows from the property (4.18) that $\mathcal{R}_t^\varepsilon H_t^\varepsilon = \mathcal{R}_t^{\varepsilon,N} H_t^\varepsilon$ and $\Pi_x^{\varepsilon,t} H_t^\varepsilon(x) = \mathcal{P}_{\varepsilon,N}(\zeta_x^{\varepsilon,t})$

(recall that $\varepsilon = 2^{-N}$), where the operator $\mathcal{P}_{\varepsilon,n}$ is defined by

$$\mathcal{P}_{\varepsilon,n}(\zeta) \stackrel{\text{def}}{=} \sum_{y \in \Lambda_n^d} \langle \zeta, \varphi_y^{N,n} \rangle_\varepsilon \, \varphi_y^{N,n} \; .$$

This allows us to choose $n_0 \geq 0$ to be the smallest integer such that $2^{-n_0} \leq \lambda$ and rewrite

$$\begin{aligned} \mathcal{R}_t^\varepsilon H_t^\varepsilon - \Pi_x^{\varepsilon,t} H_t^\varepsilon(x) &= \big(\mathcal{R}_t^{\varepsilon,n_0} H_t^\varepsilon - \mathcal{P}_{\varepsilon,n_0}(\zeta_x^{\varepsilon,t})\big) \\ &+ \sum_{n=n_0}^{N-1} \big(\mathcal{R}_t^{\varepsilon,n+1} H_t^\varepsilon - \mathcal{P}_{\varepsilon,n+1}(\zeta_x^{\varepsilon,t}) - \mathcal{R}_t^{\varepsilon,n} H_t^\varepsilon + \mathcal{P}_{\varepsilon,n}(\zeta_x^{\varepsilon,t})\big). \end{aligned} \tag{4.22}$$

The first term on the right hand side yields

$$\langle \mathcal{R}_t^{\varepsilon,n_0} H_t^\varepsilon - \mathcal{P}_{\varepsilon,n_0}(\zeta_x^{\varepsilon,t}), \varrho_x^\lambda \rangle_\varepsilon = \sum_{y \in \Lambda_{n_0}^d} \langle \zeta_{xy}^{\varepsilon,t}, \varphi_y^{N,n_0} \rangle_\varepsilon \, \langle \varphi_y^{N,n_0}, \varrho_x^\lambda \rangle_\varepsilon \; . \tag{4.23}$$

Using (4.21) and the bound $|\langle \varphi_y^{N,n_0}, \varrho_x^\lambda \rangle_\varepsilon| \lesssim 2^{n_0 d/2}$, we obtain

$$\big|\langle \mathcal{R}_t^{\varepsilon,n_0} H_t^\varepsilon - \mathcal{P}_{\varepsilon,n_0}(\zeta_x^{\varepsilon,t}), \varrho_x^\lambda \rangle_\varepsilon\big| \lesssim \|\Pi^\varepsilon\|_{\gamma;T}^{(\varepsilon)} \|H^\varepsilon\|_{\gamma,\eta;T}^{(\varepsilon)} |t|_\varepsilon^{\eta-\gamma} 2^{-\gamma n_0} \; .$$

Here, we have also used $|x - y| \lesssim 2^{-n_0}$ in the sum in (4.23), and the fact that only a finite number of points $y \in \Lambda_{n_0}^d$ contribute to this sum.

Now we will bound each term in the sum in (4.22). Using (4.19) and (4.20), we can write

$$\mathcal{R}_t^{\varepsilon,n+1} H_t^\varepsilon - \mathcal{P}_{\varepsilon,n+1}(\zeta_x^{\varepsilon,t}) - \mathcal{R}_t^{\varepsilon,n} H_t^\varepsilon + \mathcal{P}_{\varepsilon,n}(\zeta_x^{\varepsilon,t}) = g_{t,n}^\varepsilon + h_{t,n}^\varepsilon \; ,$$

where $g_{t,n}^\varepsilon$ is defined by

$$g_{t,n}^\varepsilon = \sum_{y \in \Lambda_n^d} \sum_{k \in \mathcal{K}} a_k \langle \zeta_{y,y+2^{-n}k}^{\varepsilon,t}, \varphi_{y+2^{-n}k}^{N,n+1} \rangle_\varepsilon \, \varphi_y^{N,n}$$

and the constants $\{a_k : k \in \mathcal{K}\}$ are from (4.19). For $h_{t,n}^\varepsilon$ we have the identity

$$h_{t,n}^\varepsilon = \sum_{y \in \Lambda_{n+1}^d} \sum_{k \in \mathcal{K}} \sum_{\psi \in \Psi} b_k \langle \zeta_{xy}^{\varepsilon,t}, \varphi_y^{N,n+1} \rangle_\varepsilon \, \psi_{y-2^{-n}k}^{N,n} \; . \tag{4.24}$$

Moreover, the following bounds, for $n \in [n_0, N]$, follow from the properties of the functions φ_x^n and ψ_x^n:

$$|\langle \varphi_y^{N,n}, \varrho_x^\lambda \rangle_\varepsilon| \lesssim 2^{n_0 d/2} 2^{-(n-n_0)d/2} , \quad |\langle \psi_y^{N,n}, \varrho_x^\lambda \rangle_\varepsilon| \lesssim 2^{n_0 d/2} 2^{-(n-n_0)(r+d/2)} .$$

Using them and (4.21), we obtain a bound on $g_{t,n}^\varepsilon$:

$$\begin{aligned} |\langle g_{t,n}^\varepsilon, \varrho_x^\lambda \rangle_\varepsilon| &\lesssim \sum_{y \in \Lambda_n^d} \sum_{k \in \mathcal{K}} |\langle \zeta_{y, y+2^{-n}k}^{\varepsilon,t}, \varphi_{y+2^{-n}k}^{N,n+1} \rangle_\varepsilon| \, |\langle \varphi_y^{N,n}, \varrho_x^\lambda \rangle_\varepsilon| \\ &\lesssim \|\Pi^\varepsilon\|_{\gamma;T}^{(\varepsilon)} \|H^\varepsilon\|_{\gamma,\eta;T}^{(\varepsilon)} |t|_\varepsilon^{\eta-\gamma} 2^{-\gamma n} , \end{aligned}$$

where we have used $|x - y| \lesssim 2^{-n}$ in the sum. Summing these bounds over $n \in [n_0, N]$, we obtain a bound of the required order. Similarly, we obtain the following bound on (4.24):

$$|\langle h_{t,n}^\varepsilon, \varrho_x^\lambda \rangle_\varepsilon| \lesssim \|\Pi^\varepsilon\|_{\gamma;T}^{(\varepsilon)} \|H^\varepsilon\|_{\gamma,\eta;T}^{(\varepsilon)} |t|_\varepsilon^{\eta-\gamma} 2^{-\gamma n_0} 2^{-(n-n_0)(r+\alpha)} ,$$

which gives the required bound after summing over $n \in [n_0, N]$. In this estimate we have used the fact that $|y - x| \lesssim 2^{-n_0}$ in the sum in (4.24).

The bounds (4.8) can be shown similarly to (3.18) and (3.19). □

4.2.2 Convolutions with discrete kernels

In this section we describe on the abstract level convolutions with discrete kernels. We start with a definition of the kernels we will work with.

Definition 4.2.10. We say that a function $K^\varepsilon : \mathbb{R} \times \Lambda_\varepsilon^d \to \mathbb{R}$ is regularising of order $\beta > 0$, if one can find functions $K^{(\varepsilon,n)} : \mathbb{R}^{d+1} \to \mathbb{R}$ and $\mathring{K}^\varepsilon : \mathbb{R} \times \Lambda_\varepsilon^d \to \mathbb{R}$ such that

$$K^\varepsilon = \sum_{n=0}^{N-1} K^{(\varepsilon,n)} + \mathring{K}^\varepsilon \stackrel{\text{def}}{=} \bar{K}^\varepsilon + \mathring{K}^\varepsilon , \tag{4.25}$$

where the function $K^{(\varepsilon,n)}$ has the same support and bounds as the function $K^{(n)}$ in Definition 3.2.16, for some $c, r > 0$, and furthermore, for $k \in \mathbb{N}^{d+1}$ such that $|k|_\mathfrak{s} \leq r$, it satisfies

$$\int_{\mathbb{R} \times \Lambda_\varepsilon^d} z^k K^{(\varepsilon,n)}(z) \, dz = 0 . \tag{4.26}$$

The function $\mathring{K}^\varepsilon$ is supported in $\{z \in \mathbb{R} \times \Lambda_\varepsilon^d : \|z\|_\mathfrak{s} \leq c\varepsilon\}$ and satisfies (4.26) with $k = 0$ and

$$\sup_{z \in \mathbb{R} \times \Lambda_\varepsilon^d} |\mathring{K}^\varepsilon(z)| \leq C\varepsilon^{-|\mathfrak{s}|+\beta} . \tag{4.27}$$

Now, we will define how a discrete model acts on an abstract integration map.

Definition 4.2.11. Let $\mathcal{I}$ be an abstract integration map of order β as in Definition 3.2.14 for a regularity structure $\mathscr{T} = (\mathcal{T}, \mathcal{G})$, let $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ be a discrete model, and let K^ε be regularising of order β with $r > -\lfloor \min \mathcal{A} \rfloor$. Let furthermore $\bar{K}^\varepsilon$ and $\mathring{K}^\varepsilon$ be as in (4.25). We define $\bar{\mathcal{J}}^\varepsilon$ on the grid in the same way as its continuous analogue in (3.33), but using $\bar{K}^\varepsilon$ instead of K and using the discrete objects instead of their continuous counterparts. Moreover, we define

$$\mathring{\mathcal{J}}^\varepsilon_{t,x}\tau \stackrel{\text{def}}{=} \mathbf{1} \int_\mathbb{R} \langle \Pi_x^{\varepsilon,s} \Sigma_x^{\varepsilon,st}\tau, \mathring{K}^\varepsilon_{t-s}(x - \cdot)\rangle_\varepsilon \, ds ,$$

and $\mathcal{J}^\varepsilon_{t,x} \stackrel{\text{def}}{=} \bar{\mathcal{J}}^\varepsilon_{t,x} + \mathring{\mathcal{J}}^\varepsilon_{t,x}$. We say that Z^ε realises K^ε for $\mathcal{I}$ if the identities (3.32) and (3.34) hold for the corresponding discrete objects. As before, these two identities should be thought of as providing the definitions of $\Gamma^{\varepsilon,t}_{xy}\mathcal{I}\tau$ and $\Sigma^{\varepsilon,st}_x\mathcal{I}\tau$ via $\Gamma^{\varepsilon,t}_{xy}\tau$ and $\Sigma^{\varepsilon,st}_x\tau$.

For a discrete modelled distribution H^ε, we define $\bar{\mathcal{N}}^\varepsilon_\gamma H^\varepsilon$ as in (3.36), but using the discrete objects instead of the continuous ones, and using the kernel $\bar{K}^\varepsilon$ instead of K. Furthermore, we define the term containing $\mathring{K}^\varepsilon$ by

$$\big(\mathring{\mathcal{N}}^\varepsilon_\gamma H^\varepsilon\big)_t(x) \stackrel{\text{def}}{=} \mathbf{1} \int_\mathbb{R} \langle \mathcal{R}^\varepsilon_s H^\varepsilon_s - \Pi_x^{\varepsilon,s}\Sigma_x^{\varepsilon,st} H^\varepsilon_t(x), \mathring{K}^\varepsilon_{t-s}(x - \cdot)\rangle_\varepsilon \, ds , \tag{4.28}$$

and we set $\mathcal{N}^\varepsilon_\gamma H^\varepsilon \stackrel{\text{def}}{=} \bar{\mathcal{N}}^\varepsilon_\gamma H^\varepsilon + \mathring{\mathcal{N}}^\varepsilon_\gamma H^\varepsilon$. Finally, we define the discrete analogue of (3.35) by

$$\big(\mathcal{K}^\varepsilon_\gamma H^\varepsilon\big)_t(x) \stackrel{\text{def}}{=} \mathcal{I} H^\varepsilon_t(x) + \mathcal{J}^\varepsilon_{t,x} H^\varepsilon_t(x) + \big(\mathcal{N}^\varepsilon_\gamma H^\varepsilon\big)_t(x) . \tag{4.29}$$

Our definition is consistent thanks to the following two lemmas.

Lemma 4.2.12. *In the setting of Definition 4.2.11, let* $\min \mathcal{A} + \beta > 0$. *Then all the algebraic relations of Definition 4.2.1 hold for the symbol* $\mathcal{I}\tau$. *Moreover, for* $\delta > 0$ *sufficiently small and for any* $l \in \mathcal{A}$ *and* $\tau \in \mathcal{T}_l$ *such that* $l + \beta \notin \mathbb{N}$ *and* $\|\tau\| = 1$, *one has the bounds*

$$|\langle \Pi^{\varepsilon,t}_x \mathcal{I}\tau, \varphi^\lambda_x\rangle_\varepsilon| \lesssim \lambda^{l+\beta} \|\Pi^\varepsilon\|^{(\varepsilon)}_{l;T} \|\Sigma^\varepsilon\|^{(\varepsilon)}_{l;T} \big(1 + \|\Gamma^\varepsilon\|^{(\varepsilon)}_{l;T}\big)\;, \tag{4.30}$$

$$\frac{|\langle \big(\Pi^{\varepsilon,t}_x - \Pi^{\varepsilon,s}_x\big)\mathcal{I}\tau, \varphi^\lambda_x\rangle_\varepsilon|}{\big(|t-s|^{1/\mathfrak{s}_0} \vee \varepsilon\big)^\delta} \lesssim \lambda^{l+\beta-\delta} \|\Pi^\varepsilon\|^{(\varepsilon)}_{\delta,l;T} \|\Sigma^\varepsilon\|^{(\varepsilon)}_{l;T} \big(1 + \|\Gamma^\varepsilon\|^{(\varepsilon)}_{l;T}\big)\;, \tag{4.31}$$

uniformly over ε *(see Remark 4.2.7),* $x \in \Lambda^d_\varepsilon$*,* $s, t \in [-T, T]$*,* $\lambda \in [\varepsilon, 1]$ *and* $\varphi \in \mathcal{B}^r_0(\mathbb{R}^d)$.

Proof. The algebraic properties of the models for the symbol $\mathcal{I}\tau$ follow easily from Definition 4.2.11. In order to prove (4.30), we will consider the terms in (3.32) containing $\mathring{K}^\varepsilon$ separately from the others. To this end, we define

$$\big(\mathring{\Pi}^{\varepsilon,t}_x \mathcal{I}\tau\big)(y) \stackrel{\text{def}}{=} \int_{\mathbb{R}} \langle \Pi^{\varepsilon,s}_x \Sigma^{\varepsilon,st}_x \tau, \mathring{K}^\varepsilon_{t-s}(y-\cdot) - \mathring{K}^\varepsilon_{t-s}(x-\cdot)\rangle_\varepsilon \, ds\;, \tag{4.32}$$
$$\big(\bar{\Pi}^{\varepsilon,t}_x \mathcal{I}\tau\big)(y) \stackrel{\text{def}}{=} \big(\Pi^{\varepsilon,t}_x - \mathring{\Pi}^{\varepsilon,t}_x\big)\big(\mathcal{I}\tau\big)(y)\;.$$

Furthermore, for $x, y \in \Lambda^d_\varepsilon$ we use the assumption $0^0 \stackrel{\text{def}}{=} 1$ and set

$$T^l_{xy} K^{(\varepsilon,n)}_t(\cdot) \stackrel{\text{def}}{=} K^{(\varepsilon,n)}_t(y-\cdot) - \sum_{|k|_\mathfrak{s} < l+\beta} \frac{(0, y-x)^k}{k!} D^k K^{(\varepsilon,n)}_t(x-\cdot)\;.$$

Using Definitions 4.2.1 and 4.2.10 and acting as in the proof of [Hai14, Lem. 5.19], we can obtain the following analogues of the bounds [Hai14, Eq. 5.33]:

$$\begin{aligned} |\langle \Pi^{\varepsilon,r}_x \Sigma^{\varepsilon,rt}_x \tau, T^l_{xy} K^{(\varepsilon,n)}_{t-r}\rangle_\varepsilon| &\lesssim \sum_{\zeta>0} |y-x|^{l+\beta+\zeta} 2^{(\mathfrak{s}_0+\zeta)n} \mathbf{1}_{|t-r|\lesssim 2^{-\mathfrak{s}_0 n}}\;, \\ \Big|\int_{\Lambda^d_\varepsilon} \langle \Pi^{\varepsilon,r}_x \Sigma^{\varepsilon,rt}_x \tau, T^l_{xy} K^{(\varepsilon,n)}_{t-r}\rangle_\varepsilon \, \varphi^\lambda_x(y)\, dy\Big| &\lesssim \sum_{\zeta>0} \lambda^{l+\beta-\zeta} 2^{(\mathfrak{s}_0-\zeta)n} \mathbf{1}_{|t-r|\lesssim 2^{-\mathfrak{s}_0 n}}\;, \end{aligned} \tag{4.33}$$

for $\varepsilon \leq |y-x| \leq 1$, $\lambda \in [\varepsilon, 1]$, with ζ taking a finite number of values and with the proportionality constants as in (4.30). Integrating these bounds in the time variable r and using the first bound in (4.33) in the case $|y-x| \leq 2^{-n}$ and the second bound in the case $2^{-n} \leq \lambda$, we obtain the required estimate on $\langle \bar{\Pi}^{\varepsilon,t}_x \mathcal{I}\tau, \varphi^\lambda_x\rangle_\varepsilon$.

In order to bound $\big(\bar{\Pi}^{\varepsilon,t}_x - \bar{\Pi}^{\varepsilon,s}_x\big)\mathcal{I}\tau$, we consider two cases $|t-s| \geq 2^{-\mathfrak{s}_0 n}$ and $|t-s| < 2^{-\mathfrak{s}_0 n}$. In the first case we estimate $\bar{\Pi}^{\varepsilon,t}_x \mathcal{I}\tau$ and $\bar{\Pi}^{\varepsilon,s}_x \mathcal{I}\tau$ separately using (4.33), and obtain the required bound, if $\delta > 0$ is sufficiently small. In the case $|t-s| < 2^{-\mathfrak{s}_0 n}$ we write

$$\begin{aligned}&\langle \Pi_x^{\varepsilon,r}\Sigma_x^{\varepsilon,rt}\tau, T_{xy}^l K_{t-r}^{(\varepsilon,n)}\rangle_\varepsilon - \langle \Pi_x^{\varepsilon,r}\Sigma_x^{\varepsilon,rs}\tau, T_{xy}^l K_{s-r}^{(\varepsilon,n)}\rangle_\varepsilon\\ &\quad = \langle \Pi_x^{\varepsilon,r}\Sigma_x^{\varepsilon,rs}\big(\Sigma_x^{\varepsilon,st}-1\big)\tau, T_{xy}^l K_{t-r}^{(\varepsilon,n)}\rangle_\varepsilon + \langle \Pi_x^{\varepsilon,r}\Sigma_x^{\varepsilon,rs}\tau, T_{xy}^l\big(K_{t-r}^{(\varepsilon,n)} - K_{s-r}^{(\varepsilon,n)}\big)\rangle_\varepsilon\;,\end{aligned}$$

and estimate each of these terms similarly to (4.33), which gives the required bound for sufficiently small $\delta > 0$.

It is only left to prove the required bounds for $\mathring{\Pi}_x^{\varepsilon,t}\big(\mathcal{I}\tau\big)$. It follows immediately from Definition 4.2.1 that $\big|\big(\Pi_x^{\varepsilon,t}a\big)(x)\big| \lesssim \|a\|\varepsilon^\zeta$, for $a \in \mathcal{T}_\zeta$. Hence, using the properties (3.8) and (3.9) we obtain

$$\begin{aligned}\int_{\mathbb{R}}\big|\langle \Pi_x^{\varepsilon,s}\Sigma_x^{\varepsilon,st}\tau, \mathring{K}_{t-s}^\varepsilon(y-\cdot)\rangle_\varepsilon\big|\,ds &= \int_{\mathbb{R}}\big|\langle \Pi_y^{\varepsilon,s}\Sigma_y^{\varepsilon,st}\Gamma_{yx}^{\varepsilon,t}\tau, \mathring{K}_{t-s}^\varepsilon(y-\cdot)\rangle_\varepsilon\big|\,ds\\ &\lesssim \sum_{\zeta\le l}\varepsilon^{\zeta+\beta}|y-x|^{l-\zeta}\;,\end{aligned} \tag{4.34}$$

where $\zeta \in \mathcal{A}$. Similarly, the second term in (4.32) is bounded by $\varepsilon^{l+\beta}$, implying that if $\lambda \geq \varepsilon$ and $\min \mathcal{A} + \beta > 0$, then one has

$$|\langle \mathring{\Pi}_x^{\varepsilon,t}\mathcal{I}\tau, \varphi_x^\lambda\rangle_\varepsilon| \lesssim \sum_{\zeta\le l}\varepsilon^{\zeta+\beta}\lambda^{l-\zeta} \lesssim \lambda^{l+\beta}\;, \tag{4.35}$$

which finishes the proof of (4.30). In order to complete the proof of (4.31), we use (4.34) and brutally bound

$$\begin{aligned}|\langle\big(\mathring{\Pi}_x^{\varepsilon,t} - \mathring{\Pi}_x^{\varepsilon,s}\big)\mathcal{I}\tau, \varphi_x^\lambda\rangle_\varepsilon| &\le |\langle \mathring{\Pi}_x^{\varepsilon,t}\mathcal{I}\tau, \varphi_x^\lambda\rangle_\varepsilon| + |\langle \mathring{\Pi}_x^{\varepsilon,s}\mathcal{I}\tau, \varphi_x^\lambda\rangle_\varepsilon|\\ &\lesssim \sum_{\zeta\le l}\varepsilon^{\zeta+\beta}|y-x|^{l-\zeta} \lesssim \big(|t-s|^{1/\mathfrak{s}_0}\vee\varepsilon\big)^{\tilde\delta}\sum_{\zeta\le l}\varepsilon^{\zeta+\beta-\tilde\delta}|y-x|^{l-\zeta}\;,\end{aligned}$$

from which we obtain the required bound in the same way as before, as soon as $\delta \in (0, \min\mathcal{A}+\beta)$. $\square$

The following lemma provides a relation between $\mathcal{J}^\varepsilon$ and the operators Γ^ε, Σ^ε.

Lemma 4.2.13. *In the setting of Lemma 4.2.12, the operators*

$$\mathcal{J}_{xy}^{\varepsilon,t} \stackrel{\text{def}}{=} \mathcal{J}_{t,x}^\varepsilon\Gamma_{xy}^{\varepsilon,t} - \Gamma_{xy}^{\varepsilon,t}\mathcal{J}_{t,y}^\varepsilon\;, \qquad \mathcal{J}_x^{\varepsilon,st} \stackrel{\text{def}}{=} \mathcal{J}_{s,x}^\varepsilon\Sigma_x^{\varepsilon,st} - \Sigma_x^{\varepsilon,st}\mathcal{J}_{t,x}^\varepsilon\;, \tag{4.36}$$

with $s, t \in \mathbb{R}$ and $x, y \in \Lambda_\varepsilon^d$, satisfy the following bounds:

$$\begin{aligned}\big|\big(\mathcal{J}_{xy}^{\varepsilon,t}\tau\big)_k\big| &\lesssim \|\Pi^\varepsilon\|_{l;T}^{(\varepsilon)}\|\Sigma^\varepsilon\|_{l;T}^{(\varepsilon)}\big(1+\|\Gamma^\varepsilon\|_{l;T}^{(\varepsilon)}\big)|x-y|^{l+\beta-|k|_\mathfrak{s}}\;,\\ \big|\big(\mathcal{J}_{x}^{\varepsilon,st}\tau\big)_k\big| &\lesssim \|\Pi^\varepsilon\|_{l;T}^{(\varepsilon)}\|\Sigma^\varepsilon\|_{l;T}^{(\varepsilon)}\big(1+\|\Gamma^\varepsilon\|_{l;T}^{(\varepsilon)}\big)\big(|t-s|^{1/\mathfrak{s}_0}\vee\varepsilon\big)^{l+\beta-|k|_\mathfrak{s}}\;,\end{aligned} \tag{4.37}$$

uniformly in ε (see Remark 4.2.7), for τ as in Lemma 4.2.12, for any $k \in \mathbb{N}^{d+1}$ such that $|k|_\mathfrak{s} < l + \beta$, and for $(\cdot)_k$ being the multiplier of X^k. In particular, the required bounds on $\Gamma^\varepsilon\mathcal{I}\tau$ and $\Sigma^\varepsilon\mathcal{I}\tau$ from Definition 4.2.1 hold.

Proof. The bounds on the parts of $\mathcal{J}_{xy}^{\varepsilon,t}\tau$ and $\mathcal{J}_{x}^{\varepsilon,st}\tau$ not containing $\mathring{K}^\varepsilon$ can be obtained as in [Hai14, Lem. 5.21], where the bound on the right-hand side of (4.37) comes from the fact that the scaling of the kernels $K^{(\varepsilon,n)}$ in (4.25) does not go below ε. The contributions to (4.36) from the kernel $\mathring{K}^\varepsilon$ come via the terms $\mathring{\mathcal{J}}_{t,x}^\varepsilon\Gamma_{xy}^{\varepsilon,t}$, $\mathring{\mathcal{J}}_{t,y}^\varepsilon$, $\mathring{\mathcal{J}}_{s,x}^\varepsilon\Sigma_x^{\varepsilon,st}$ and $\mathring{\mathcal{J}}_{t,x}^\varepsilon$. We can bound all of them separately, similarly to (4.34), and use $|x-y| \geq \varepsilon$ and $|t-s|^{1/\mathfrak{s}_0} \vee \varepsilon \geq \varepsilon$ to estimate the powers of ε. Since all of these powers are positive by assumption, this yields the required bounds.

Now, we will prove the bound on $\Gamma^\varepsilon\mathcal{I}\tau$ required by Definition 4.2.1. For $m < l$ such that $m \notin \mathbb{N}$, (3.34) yields

$$\|\Gamma_{xy}^{\varepsilon,t}\mathcal{I}\tau\|_m = \|\mathcal{I}\big(\Gamma_{xy}^{\varepsilon,t}\tau\big)\|_m \leq \|\Gamma_{xy}^{\varepsilon,t}\tau\|_{m-\beta} \lesssim |y-x|^{l+\beta-m}\;,$$

where we have used the properties of $\mathcal{I}$. Similarly, we can bound $\|\Sigma_x^{\varepsilon,st}\mathcal{I}\tau\|_m$. Furthermore, since the map $\mathcal{I}$ does not produce elements of integer homogeneity, we have for $m \in \mathbb{N}$,

$$\|\Gamma_{xy}^{\varepsilon,t}\mathcal{I}\tau\|_m = \|\mathcal{J}_{xy}^{\varepsilon,t}\|_m \lesssim |y-x|^{l+\beta-m}\;,$$

where the last bound we have proved above. In the same way we can obtain the required bound on $\|\Sigma_x^{\varepsilon,st}\mathcal{I}\tau\|_m$. $\square$

Remark 4.2.14. If $(\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ is a discrete model on $\mathscr{T}^{\mathrm{gen}}$, which is introduced in Definition 3.3.5, then there is a canonical way to extend it to a discrete model on $\hat{\mathscr{T}}$. Since the symbols from $\hat{\mathcal{F}}$ are "generated" by $\mathcal{F}^{\mathrm{gen}}$, we only have to define the actions of Π^ε, Γ^ε and Σ^ε on the symbols $\tau\bar{\tau}$ and $\mathcal{I}\tau \in \hat{\mathcal{F}} \setminus \mathcal{F}^{\mathrm{gen}}$ with $\tau, \bar{\tau} \in \hat{\mathcal{F}}$, so that the extension of the model to $\hat{\mathscr{T}}$ will follow by induction. For the product $\tau\bar{\tau}$, we set

$$\left(\Pi_x^{\varepsilon,t}\tau\bar{\tau}\right)(y) = \left(\Pi_x^{\varepsilon,t}\tau\right)(y)\left(\Pi_x^{\varepsilon,t}\bar{\tau}\right)(y)\;, \tag{4.38a}$$

$$\Sigma_x^{\varepsilon,st}\tau\bar{\tau} = \left(\Sigma_x^{\varepsilon,st}\tau\right)\left(\Sigma_x^{\varepsilon,st}\bar{\tau}\right)\;, \qquad \Gamma_{xy}^{\varepsilon,t}\tau\bar{\tau} = \left(\Gamma_{xy}^{\varepsilon,t}\tau\right)\left(\Gamma_{xy}^{\varepsilon,t}\bar{\tau}\right)\;. \tag{4.38b}$$

For the symbol $\mathcal{I}\tau$ we define the actions of the maps $(\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ by the identities (3.32) and (3.34). However, even if the family of models satisfy analytic bounds uniformly in ε on $\mathscr{T}^{\mathrm{gen}}$, this is not necessarily true for its extension to $\hat{\mathscr{T}}$.

The structure of the canonical extension of a discrete model will be important for us. That is why we make the following definition.

Definition 4.2.15. We call a discrete model $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ defined on $\hat{\mathscr{T}}$ *admissible*, if it satisfies the identities (4.38b) and furthermore realises K^ε for $\mathcal{I}$.

Remark 4.2.16. If $M \in \mathfrak{R}$ is a renormalisation map as mentioned in Section 3.3.1, such that $M\hat{\mathcal{T}} \subset \hat{\mathcal{T}}$, where $\hat{\mathcal{T}}$ is introduced in Definition 3.3.5, and if $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ is an admissible model, then we can define a renormalised discrete model $\hat{Z}^\varepsilon$ as in [Hai14, Sec. 8.3], which is also admissible.

The following result is a discrete analogue of Theorem 3.2.21.

Theorem 4.2.17. *For a regularity structure $\mathscr{T} = (\mathcal{T}, \mathcal{G})$ with the minimal homogeneity α, let β, γ, η, $\bar{\gamma}$, $\bar{\eta}$ and r be as in Theorem 3.2.21 and let $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ be a discrete model which realises K^ε for $\mathcal{I}$. Then for any discrete modelled distribution H^ε the following bound holds*

$$\|\!|\mathcal{K}_\gamma^\varepsilon H^\varepsilon|\!\|_{\bar{\gamma},\bar{\eta};T}^{(\varepsilon)} \lesssim \|\!|H^\varepsilon|\!\|_{\gamma,\eta;T}^{(\varepsilon)} \|\Pi^\varepsilon\|_{\gamma;T}^{(\varepsilon)} \|\Sigma^\varepsilon\|_{\gamma;T}^{(\varepsilon)} \big(1 + \|\Gamma^\varepsilon\|_{\bar{\gamma};T}^{(\varepsilon)} + \|\Sigma^\varepsilon\|_{\bar{\gamma};T}^{(\varepsilon)}\big)\;, \tag{4.39}$$

uniformly in ε (see Remark 4.2.7), and one has the identity

$$\mathcal{R}_t^\varepsilon \big(\mathcal{K}_\gamma^\varepsilon H^\varepsilon\big)_t(x) = \int_0^t \langle \mathcal{R}_s^\varepsilon H_s^\varepsilon, K_{t-s}^\varepsilon(x - \cdot)\rangle_\varepsilon \, ds\;. \tag{4.40}$$

Moreover, if $\bar{Z}^\varepsilon = (\bar{\Pi}^\varepsilon, \bar{\Gamma}^\varepsilon, \bar{\Sigma}^\varepsilon)$ is another discrete model realising K^ε for $\mathcal{I}$, and if $\bar{\mathcal{K}}_\gamma^\varepsilon$ is defined as in (4.29) *for this model, then one has the bound*

$$\|\!|\mathcal{K}_\gamma^\varepsilon H^\varepsilon; \bar{\mathcal{K}}_\gamma^\varepsilon \bar{H}^\varepsilon|\!\|_{\bar{\gamma},\bar{\eta};T}^{(\varepsilon)} \lesssim \|\!|H^\varepsilon; \bar{H}^\varepsilon|\!\|_{\gamma,\eta;T}^{(\varepsilon)} + \|\!|Z^\varepsilon; \bar{Z}^\varepsilon|\!\|_{\bar{\gamma};T}^{(\varepsilon)}\;, \tag{4.41}$$

for all discrete modelled distributions H^ε and $\bar{H}^\varepsilon$, where the norms on H^ε and $\bar{H}^\varepsilon$ are defined via the models Z^ε and $\bar{Z}^\varepsilon$ respectively, and the proportionality constant depends on ε only via the same norms of the discrete objects as in (3.39).

Proof. The proof of the bound (4.39) for the components of $\mathcal{K}^\varepsilon_\gamma H^\varepsilon$ not containing $\mathring{K}^\varepsilon$ is almost identical to that of (3.37), and we only need to bound the terms $\mathring{\mathcal{J}}^\varepsilon H^\varepsilon$ and $\mathring{\mathcal{N}}^\varepsilon_\gamma H^\varepsilon$. The estimates on $\mathring{\mathcal{J}}^\varepsilon H^\varepsilon$ were obtained in the proof of Lemma 4.2.13. To bound $\mathring{\mathcal{N}}^\varepsilon_\gamma H^\varepsilon$, for $x, y \in \Lambda^d_\varepsilon$, we write

$$\begin{aligned}\big(\mathcal{R}^\varepsilon_s H^\varepsilon_s - \Pi^{\varepsilon,s}_x \Sigma^{\varepsilon,st}_x H^\varepsilon_t(x)\big)(y) = \Pi^{\varepsilon,s}_y \big(H^\varepsilon_s(y) - \Gamma^{\varepsilon,s}_{yx} H^\varepsilon_s(x)\big)(y)\\ + \Pi^{\varepsilon,s}_y \Gamma^{\varepsilon,s}_{yx} \big(H^\varepsilon_s(x) - \Sigma^{\varepsilon,st}_x H^\varepsilon_t(x)\big)(y) \;,\end{aligned}$$

where we made use of Definitions 4.2.5 and 4.2.1. Estimating this expression similarly to (4.34), but using (4.6) this time, we obtain

$$\|\big(\mathring{\mathcal{N}}^\varepsilon_\gamma H^\varepsilon\big)_t(x)\|_0 \lesssim |t|^{\eta-\gamma}_\varepsilon \varepsilon^{\gamma+\beta} \lesssim |t|^{\eta+\beta}_\varepsilon \;, \tag{4.42}$$

where we have used $\gamma + \beta > 0$.

Furthermore, the operator $\Gamma^{\varepsilon,t}_{yx}$ leaves $\mathbf{1}$ invariant, and we have

$$\Gamma^{\varepsilon,t}_{yx}\big(\mathring{\mathcal{N}}^\varepsilon_\gamma H^\varepsilon\big)_t(x) = \big(\mathring{\mathcal{N}}^\varepsilon_\gamma H^\varepsilon\big)_t(x) \;.$$

Thus, estimating $\big(\mathring{\mathcal{N}}^\varepsilon_\gamma H^\varepsilon\big)_t(y)$ and $\big(\mathring{\mathcal{N}}^\varepsilon_\gamma H^\varepsilon\big)_t(x)$ separately by the intermediate bound in (4.42) and using $|x-y| \geq \varepsilon$, yields the required bound. In the same way we obtain the required estimate on $\Sigma^{\varepsilon,st}_x\big(\mathring{\mathcal{N}}^\varepsilon_\gamma H^\varepsilon\big)_t(x) - \big(\mathring{\mathcal{N}}^\varepsilon_\gamma H^\varepsilon\big)_s(x)$.

The bound (4.41) can be show similarly to (3.39), using the above approach. In order to show that the identity (4.40) holds, we notice that

$$\big(\mathcal{K}^\varepsilon_\gamma H^\varepsilon\big)_t(x) \in \mathcal{T}_{\mathrm{poly}} + \mathcal{T}_{\geq \alpha+\beta} \;,$$

where $\mathcal{T}_{\mathrm{poly}}$ contains only the abstract polynomials and $\alpha + \beta > 0$ by assumption. It hence follows from Definitions 4.2.1 and 4.2.5 that

$$\mathcal{R}^\varepsilon_t\big(\mathcal{K}^\varepsilon_\gamma H^\varepsilon\big)_t(x) = \langle \mathbf{1}, \big(\mathcal{K}^\varepsilon_\gamma H^\varepsilon\big)_t(x)\rangle \;,$$

which is equal to the right-hand side of (4.40). $\square$

4.3 Analysis of discrete stochastic PDEs

We consider the following spatial discretisation of equation (3.40) on $\mathbb{R}_+ \times \Lambda_\varepsilon^d$:

$$\partial_t u^\varepsilon = A^\varepsilon u^\varepsilon + F^\varepsilon(u^\varepsilon, \xi^\varepsilon) , \qquad u^\varepsilon(0, \cdot) = u_0^\varepsilon(\cdot) , \tag{4.43}$$

where $u_0^\varepsilon \in \mathbb{R}^{\Lambda_\varepsilon^d}$, ξ^ε is a spatial discretisation of ξ, F^ε is a discrete approximation of F, and $A^\varepsilon : \ell^\infty(\Lambda_\varepsilon^d) \to \ell^\infty(\Lambda_\varepsilon^d)$ is a bounded linear operator satisfying the following assumption.

Assumption 4.3.1. *There exists an operator A given by a Fourier multiplier $a : \mathbb{R}^d \to \mathbb{R}$ satisfying Assumption 3.3.1 with an even integer parameter $\beta > 0$ and a measure μ on $\mathbb{Z}^d$ with finite support such that*

$$\big(A^\varepsilon \varphi\big)(x) = \varepsilon^{-\beta} \int_{\mathbb{R}^d} \varphi(x - \varepsilon y)\, \mu(dy) , \qquad x \in \Lambda_\varepsilon^d , \tag{4.44}$$

for every $\varphi \in \mathcal{C}(\mathbb{R}^d)$, and such that the identity

$$\int_{\mathbb{R}^d} P(x-y)\, \mu(dy) = (AP)(x) , \qquad x \in \mathbb{R}^d , \tag{4.45}$$

holds for every polynomial P on $\mathbb{R}^d$ with $\deg P \leq \beta$. Furthermore, the Fourier transform of μ only vanishes on $\mathbb{Z}^d$.

Example 4.3.2. A common example of the operator A is the Laplacian Δ, with its nearest neighbor discrete approximation Δ^ε, defined by (4.44) with the measure μ given by

$$\mu(\varphi) = \sum_{x \in \mathbb{Z}^d : \|x\| = 1} \big(\varphi(x) - \varphi(0)\big) , \tag{4.46}$$

for every $\varphi \in \ell^\infty(\mathbb{Z}^d)$, and where $\|x\|$ is the Euclidean norm. In this case, the Fourier multiplier of Δ is $a(\zeta) = -4\pi^2 \|\zeta\|^2$ and

$$\big(\mathscr{F}\mu\big)(\zeta) = -4 \sum_{i=1}^{d} \sin^2\big(\pi \zeta_i\big) , \qquad \zeta \in \mathbb{R}^d ,$$

where $\mathscr{F}$ is the Fourier transform. One can see that Assumption 4.3.1 is satisfied with $\beta = 2$.

The following section is devoted to the analysis of discrete operators.

4.3.1 Analysis of discrete operators

We assume that the operator $A^\varepsilon : \ell^\infty(\Lambda_\varepsilon^d) \to \ell^\infty(\Lambda_\varepsilon^d)$ satisfies Assumption 4.3.1 and we define the Green's function of $\partial_t - A^\varepsilon$ by

$$G_t^\varepsilon(x) \stackrel{\text{def}}{=} \varepsilon^{-d} \mathbf{1}_{t \geq 0} \big(e^{tA^\varepsilon} \delta_{0,\cdot}\big)(x) \;, \qquad (t, x) \in \mathbb{R} \times \Lambda_\varepsilon^d \;, \tag{4.47}$$

where $\delta_{\cdot,\cdot}$ is the Kronecker's delta.

In order to build an extension of G^ε off the grid, we first choose a function $\varphi \in \mathcal{S}(\mathbb{R}^d)$ whose values coincide with $\delta_{0,\cdot}$ on $\mathbb{Z}^d$, and such that $\big(\mathscr{F}\varphi\big)(\zeta) = 0$ for $|\zeta|_\infty \geq 3/4$, say, where $\mathscr{F}$ is the Fourier transform. To build such a function, write $\tilde\varphi \in \mathcal{C}^\infty(\mathbb{R}^d)$ for the Dirichlet kernel $\tilde\varphi(x) = \prod_{i=1}^d \frac{\sin(\pi x_i)}{\pi x_i}$, whose values coincide with $\delta_{0,x}$ for $x \in \mathbb{Z}^d$, and whose Fourier transform is supported in $\{\zeta : |\zeta|_\infty \leq \frac{1}{2}\}$. Choosing any function $\psi \in \mathcal{C}^\infty(\mathbb{R}^d)$ supported in the ball of radius $1/4$ around the origin and integrating to 1, it then suffices to set $\mathscr{F}\varphi = \big(\mathscr{F}\tilde\varphi\big) * \psi$.

Furthermore, we define the bounded operator $\tilde A^\varepsilon : \mathcal{C}_b(\mathbb{R}^d) \to \mathcal{C}_b(\mathbb{R}^d)$ by the right-hand side of (4.44), where $\mathcal{C}_b(\mathbb{R}^d)$ is the space of bounded continuous functions on $\mathbb{R}^d$ equipped with the supremum norm. Then, denoting as usual by φ^ε the rescaled version of φ, we have for G^ε the representation

$$G_t^\varepsilon(x) = \mathbf{1}_{t \geq 0} \big(e^{t\tilde A^\varepsilon} \varphi^\varepsilon\big)(x) \;, \qquad (t, x) \in \mathbb{R} \times \Lambda_\varepsilon^d \;. \tag{4.48}$$

By setting $x \in \mathbb{R}^d$ in (4.48), we obtain an extension of G^ε to $\mathbb{R}^{d+1}$, which we again denote by G^ε.

Unfortunately, the function $G_t^\varepsilon(x)$ is discontinuous at $t = 0$, and our next aim is to modify it in such a way that it becomes differentiable at least for sufficiently large values of $|x|$. Since $\tilde A^\varepsilon$ generates a strongly continuous semigroup, for every $m \in \mathbb{N}$ we have the uniform limit

$$\lim_{t \downarrow 0} \partial_t^m G_t^\varepsilon = \big(\tilde A^\varepsilon\big)^m \varphi^\varepsilon \;. \tag{4.49}$$

This gives us the terms which we have to subtract from G^ε to make it continuously differentiable at $t = 0$. For this, we take a function $\varrho : \mathbb{R} \to \mathbb{R}$ such that

$$\varrho(t) = \begin{cases} 1\,, & \text{for} \quad t \in \left[0, \frac{1}{2}\right]\,, \\ 0\,, & \text{for} \quad t \in (-\infty, 0) \cup [1, +\infty)\,, \end{cases}$$

and $\varrho(t)$ is smooth on $t > 0$. Then, for $r > 0$, we define

$$T^{\varepsilon,r}(t,x) \stackrel{\text{def}}{=} \varrho\bigl(t/\varepsilon^{\beta}\bigr) \sum_{m \le r/\beta} \frac{t^m}{m!} \bigl(\tilde{A}^{\varepsilon}\bigr)^m \varphi^{\varepsilon}(x)\,, \qquad (t,x) \in \mathbb{R}^{d+1}\,. \tag{4.50}$$

The role of the function ϱ is to have $T^{\varepsilon,r}$ compactly supported in t. Then we have the following result.

Lemma 4.3.3. *In the described context, let Assumption 4.3.1 be satisfied. Then for every fixed value $r > 0$ there exists a constant $c > 0$ such that the bound*

$$\bigl|D^k\bigl(G^{\varepsilon} - T^{\varepsilon,r}\bigr)(z)\bigr| \le C\|z\|_{\mathfrak{s}}^{-d-|k|_{\mathfrak{s}}}\,, \tag{4.51}$$

holds uniformly over $z \in \mathbb{R}^{d+1}$ with $\|z\|_{\mathfrak{s}} \ge c\varepsilon$, for all $k \in \mathbb{N}^{d+1}$ with $|k|_{\mathfrak{s}} \le r$, for D^k begin a space-time derivative and for the space-time scaling $\mathfrak{s} = (\beta, 1, \ldots, 1)$.

Moreover, for $|t|_{\varepsilon} \stackrel{def}{=} |t|^{1/\beta} \vee \varepsilon$, the function $\bar{G}^{\varepsilon}_t(x) \stackrel{def}{=} |t|_{\varepsilon}^d G^{\varepsilon}_t\bigl(|t|_{\varepsilon} x\bigr)$ is Schwartz in x, i.e. for every $m \in \mathbb{N}$ and $\bar{k} \in \mathbb{N}^d$ there is a constant $\bar{C}$ such that the bound

$$\bigl|D^{\bar{k}}_x \bar{G}^{\varepsilon}_t(x)\bigr| \le \bar{C}\bigl(1 + |x|\bigr)^{-m}\,, \tag{4.52}$$

holds uniformly over $(t,x) \in \mathbb{R}^{d+1}$.

Proof. The function $G^{\varepsilon} - T^{\varepsilon,r}$ is of class $\mathcal{C}^r_{\mathfrak{s}}$ on $\mathbb{R}^{d+1}$. Indeed, spatial regularity follows immediately from the regularity of φ and commutation of $\tilde{A}^{\varepsilon}$ with the differential operator. Continuous differentiability at $t = 0$ follows from (4.49). Furthermore, since G^{ε} vanishes on $t \le 0$, we only need to consider $t > 0$.

Next, we notice that the bound (4.51) follows from (4.52). Let $\hat{r} > 0$ be such that the measure μ in Assumption 4.3.1 is supported in the ball of radius $\hat{r}$. Then, for $k = (k_0, \bar{k}) \in \mathbb{N}^{d+1}$ with $k_0 \in \mathbb{N}$ and $|k|_{\mathfrak{s}} \le r$ we use (4.48) and the identities (4.45), combined with the Taylor's formula, to get

$$\bigl|D^k G^{\varepsilon}_t(x)\bigr| = \bigl|\bigl(\tilde{A}^{\varepsilon}\bigr)^{k_0} D^{\bar{k}}_x G^{\varepsilon}_t(x)\bigr| \lesssim \sup_{y : |y-x| \le k_0 \hat{r}\varepsilon} \sup_{l : |l| = \beta k_0} \bigl|D^{\bar{k}+l}_y G^{\varepsilon}_t(y)\bigr|\,, \tag{4.53}$$

where $y \in \mathbb{R}^d$, $l \in \mathbb{N}^d$. For $\|t,x\|_\mathfrak{s} \geq c\varepsilon$, in the case $|t|^{1/\beta} \geq |x|$, we bound the right-hand side of (4.53) using (4.52) with $m = 0$, what gives an estimate of order $|t|^{-(d+|k|_\mathfrak{s})/\beta}$. In the case $|t|^{1/\beta} < |x|$, we use (4.52) with $m = d + |k|_s$, and we get a bound of order $|x|^{-d-|k|_\mathfrak{s}}$, if we take $c \geq 2r\hat{r}/\beta$. Furthermore, the required bound on $T^{\varepsilon,r}$ follows easily from the properties of the functions φ and ϱ. Hence, we only need to prove the bound (4.52).

Denoting by $\mathscr{F}$ the Fourier transform, we get from (4.48) and Assumption 4.3.1:

$$\big(\mathscr{F}\bar{G}^\varepsilon_t\big)(\zeta) = \big(\mathscr{F}\varphi\big)\big(\varepsilon|t|_\varepsilon^{-1}\zeta\big)\, e^{t|t|_\varepsilon^{-1}a(\zeta)f(\varepsilon|t|_\varepsilon^{-1}\zeta)}\;, \tag{4.54}$$

where we have used the scaling property $\lambda^\beta a(\zeta) = a(\lambda\zeta)$, and where $f \stackrel{\text{def}}{=} (\mathscr{F}\mu)/a$.

We start with considering the case $t \geq \varepsilon^\beta$. It follows from the last part of Assumption 4.3.1 that there exists $\bar{c} > 0$ such that $f(\zeta) \geq \bar{c}$ for $|\zeta|_\infty \leq 3/4$. Since $\varepsilon|t|_\varepsilon^{-1} \leq 1$, we conclude that

$$\big|D_\zeta^{\bar{k}} e^{a(\zeta)f(\varepsilon|t|_\varepsilon^{-1}\zeta)}\big| \lesssim |\zeta|^{\beta|\bar{k}|} e^{a(\zeta)\bar{c}} \lesssim \big(1+|\zeta|\big)^{-m}\;,$$

for $|\zeta|_\infty < 3/\big(4\varepsilon|t|_\varepsilon^{-1}\big)$, for every $m \geq 0$ and for a proportionality constant dependent on m and $\bar{k}$. Here, we have used $a(\zeta) < 0$ and polynomial growth of $|a(\zeta)|$. Since $\big(\mathscr{F}\varphi\big)\big(\varepsilon|t|_\varepsilon^{-1}\zeta\big)$ vanishes for $|\zeta|_\infty \geq 3/\big(4\varepsilon|t|_\varepsilon^{-1}\big)$, we conclude that

$$\big|D_\zeta^{\bar{k}}\big(\mathscr{F}\bar{G}^\varepsilon_t\big)(\zeta)\big| \lesssim \big(1+|\zeta|\big)^{-m}\;,$$

uniformly in t and ε (provided that $t \geq \varepsilon^\beta$), and for every $m \in \mathbb{N}$ and $\bar{k} \in \mathbb{N}^d$.

In the case $t < \varepsilon^\beta$, we can bound the exponent in (4.54) by 1, and the polynomial decay comes from the factor $\big(\mathscr{F}\varphi\big)\big(\zeta\big)$, because $\varphi \in \mathcal{S}(\mathbb{R}^d)$. Since the Fourier transform is continuous on Schwartz space, this implies that $\bar{G}^\varepsilon_t$ is a Schwartz function, with bounds uniform in ε and t, which is exactly the claim. $\square$

The following result is an analogue of Lemma 3.3.3 for G^ε.

Lemma 4.3.4. *Let Assumption 4.3.1 be satisfied. Then, the function G^ε defined in* (4.48) *can be written as $G^\varepsilon = K^\varepsilon + R^\varepsilon$ in such a way that the identity*

$$\big(G^\varepsilon \star_\varepsilon u\big)(z) = \big(K^\varepsilon \star_\varepsilon u\big)(z) + \big(R^\varepsilon \star_\varepsilon u\big)(z)\;, \tag{4.55}$$

holds for all $z \in (-\infty, 1] \times \Lambda_\varepsilon^d$ and all functions u on $\mathbb{R}_+ \times \Lambda_\varepsilon^d$, periodic in the spatial variable with some fixed period. Furthermore, K^ε is regularising of order β in the sense of Definition 4.2.10, for arbitrary (but fixed) r and with the scaling $\mathfrak{s} = (\beta, 1, \dots, 1)$. The function R^ε is compactly supported, non-anticipative and the norm $\|R^\varepsilon\|_{\mathcal{C}^r}$ is bounded uniformly in ε.

Proof. Let $M : \mathbb{R}^{d+1} \to \mathbb{R}_+$ be a smooth norm for the scaling $\mathfrak{s}$ (see for example [Hai14, Rem. 2.13]). Furthermore, let $\bar{\varrho} : \mathbb{R}_+ \to [0, 1]$ be a smooth "cutoff function" such that $\bar{\varrho}(s) = 0$ if $s \notin [1/2, 2]$, and such that $\sum_{n\in\mathbb{Z}} \bar{\varrho}(2^n s) = 1$ for all $s > 0$ (see the construction of the partition of unity in [BCD11]). For integers $n \in [0, N)$ we set the functions

$$\bar{\varrho}_n(z) \stackrel{\text{def}}{=} \bar{\varrho}(2^n M(z)) \;, \qquad \bar{\varrho}_{<0} \stackrel{\text{def}}{=} \sum_{n<0} \bar{\varrho}_n \;, \qquad \bar{\varrho}_{\geq N} \stackrel{\text{def}}{=} \sum_{n\geq N} \bar{\varrho}_n \;,$$

as well as

$$\begin{aligned} \bar{K}^{(\varepsilon,n)}(z) = \bar{\varrho}_n(z)\big(G^\varepsilon - T^{\varepsilon,r}\big)(z) \;, \qquad \bar{R}^\varepsilon(z) = \bar{\varrho}_{<0}(z)\big(G^\varepsilon - T^{\varepsilon,r}\big)(z) \;, \\ \tilde{K}^\varepsilon(z) = \bar{\varrho}_{\geq N}(z)\big(G^\varepsilon - T^{\varepsilon,r}\big)(z) + T^{\varepsilon,r}(z) \;. \end{aligned} \tag{4.56}$$

Then it follows immediately from the properties of $\bar{\varrho}$ that

$$G^\varepsilon = \sum_{n=0}^{N-1} \bar{K}^{(\varepsilon,n)} + \tilde{K}^\varepsilon + \bar{R}^\varepsilon \;.$$

Since $\bar{\varrho}_{<0}$ is supported away from the origin, we use (4.51) and Assumption 4.3.1 to conclude that $\|\bar{R}^\varepsilon\|_{\mathcal{C}^r}$ is bounded uniformly in ε. Actually, its value and derivatives even decay faster than any power of x.

Furthermore, the function $\bar{K}^{(\varepsilon,n)}$ is supported in the ball of radius $c2^{-n}$, for c as in Lemma 4.3.3, provided that the norm M was chosen such that $M(z) \geq 2c\|z\|_\mathfrak{s}$. By the same reason, the first term in (4.56) is supported in the ball of radius $c\varepsilon$. Moreover, the support property of the measure μ and the properties of the functions ϱ and φ^ε in (4.50) yield that the restriction of $T^{\varepsilon,r}$ to the grid Λ_ε^d in space is supported in the ball of radius $c\varepsilon$, as soon as $c \geq 2r\hat{r}/\beta$, where $\hat{r}$ is the support radius of the measure μ from Assumption 4.3.1.

As a consequence of (4.44), (4.48) and (4.50), we get for $0 \leq n < N$ the exact scaling properties

$$\bar{K}^{(\varepsilon,n)}(z) = 2^{nd}\bar{K}^{(\varepsilon 2^n,0)}\big(2^{\mathfrak{s}n}z\big)\;, \qquad \tilde{K}^{\varepsilon}(z) = \varepsilon^{-d}\tilde{K}^{1}\big(\varepsilon^{-\mathfrak{s}n}z\big)\;,$$

and (3.30) and (4.27) follow immediately from (4.51) and (4.50).

It remains to modify these functions in such a way that they "kill" polynomials in the sense of (4.26). To this end, we take a smooth function $P^{(N)}$ on $\mathbb{R}^{d+1}$, whose support coincides with the support of $\tilde{K}^{\varepsilon}$, which satisfies $|P^{(N)}(z)| \lesssim \varepsilon^{-d}$, for every $z \in \mathbb{R}^{d+1}$, and such that one has

$$\int_{\mathbb{R}\times\Lambda_\varepsilon^d} \big(\tilde{K}^{\varepsilon} - P^{(N)}\big)(z)\,dz = 0\;. \tag{4.57}$$

Then we define $\mathring{K}^{\varepsilon}$ to be the restriction of $\tilde{K}^{\varepsilon} - P^{(N)}$ to the grid Λ_ε^d in space. Apparently, the function $\mathring{K}^{\varepsilon}$ has the same scaling and support properties as $\tilde{K}^{\varepsilon}$, and it follows from (4.57) that it satisfies (4.26) with $k = 0$.

Moreover, we can recursively build a sequence of smooth functions $P^{(n)}$, for integers $n \in [0, N)$, such that $P^{(n)}$ in supported in the ball of radius $c2^{-n}$, the function $P^{(n)}$ satisfies the bounds in (3.30), and for every $k \in \mathbb{N}^{d+1}$ with $|k|_{\mathfrak{s}} \leq r$ one has

$$\int_{\mathbb{R}\times\Lambda_\varepsilon^d} z^k\big(\bar{K}^{(\varepsilon,n)} - P^{(n)} + P^{(n+1)}\big)(z)\,dz = 0\;. \tag{4.58}$$

Then, for such values of n, we define

$$K^{(\varepsilon,n)} = \bar{K}^{(\varepsilon,n)} - P^{(n)} + P^{(n+1)}\;, \qquad R^{\varepsilon} \stackrel{\text{def}}{=} \bar{R}^{\varepsilon} + P^{(0)}\;.$$

It follows from the properties of the functions $P^{(n)}$ that $K^{(\varepsilon,n)}$ has all the required properties. The function R^{ε} also has the required properties, and the decompositions (4.25) and (4.55) hold by construction. Finally, using (4.52), we can make the function R^{ε} compactly supported in the same way as in [Hai14, Lem. 7.7]. $\square$

Remark 4.3.5. One can see from the proof of Lemma 4.3.4 that the function $\mathring{K}^{\varepsilon}$ is $(r/\mathfrak{s}_0)$-times continuously differentiable in the time variable for $t \neq 0$ and has a discontinuity at $t = 0$.

By analogy with (3.41), we use the function R^{ε} from Lemma 4.3.4 to define for periodic $\zeta_t \in \mathbb{R}^{\Lambda_\varepsilon^d}$, $t \in \mathbb{R}$, the abstract polynomial

$$\big(R_\gamma^\varepsilon \zeta\big)_t(x) \stackrel{\text{def}}{=} \sum_{|k|_\mathfrak{s} < \gamma} \frac{X^k}{k!} \int_\mathbb{R} \langle \zeta_s, D^k R_{t-s}^\varepsilon(x - \cdot)\rangle_\varepsilon \, ds \;, \tag{4.59}$$

where as before $k \in \mathbb{N}^{d+1}$ and the mixed derivative D^k is in space-time.

4.3.2 Properties of the discrete equations

In this section we show that a discrete analogue of Theorem 3.3.11 holds for the solution map of the equation (4.43) with an operator A^ε satisfying Assumption 4.3.1.

Similarly to [Hai14, Lem. 7.5], but using the properties of G^ε proved in the previous section, we can show that for every periodic $u_0^\varepsilon \in \mathbb{R}^{\Lambda_\varepsilon^d}$, we have a discrete analogue of Lemma 3.3.7 for the map $(t, x) \mapsto S_t^\varepsilon u_0^\varepsilon(x)$, where S^ε is the semigroup generated by A^ε.

For the regularity structure $\mathscr{T}$ from Section 3.3.1, we take a truncated regularity structure $\hat{\mathscr{T}} = (\hat{\mathcal{T}}, \mathcal{G})$ and make the following assumption on the nonlinearity F^ε.

Assumption 4.3.6. *For some $0 < \bar{\gamma} \leq \gamma$, $\eta \in \mathbb{R}$, every $\varepsilon > 0$ and every discrete model Z^ε on $\hat{\mathscr{T}}$, there exist discrete modelled distributions $F_0^\varepsilon(Z^\varepsilon)$ and $I_0^\varepsilon(Z^\varepsilon)$, with exactly the same properties as of F_0 and I_0 in Assumption 3.3.9 on the grid. Furthermore, we define $\hat{F}^\varepsilon$ as in* (3.48)*, but via F^ε and F_0^ε, and we define $\hat{F}^\varepsilon(H^\varepsilon)$ for $H^\varepsilon : \mathbb{R}_+ \times \Lambda_\varepsilon^d \to \mathcal{T}_{<\gamma}$ as in* (3.49)*. Finally, we assume that the discrete analogue of the Lipschitz condition* (3.51) *holds for $\hat{F}^\varepsilon$, with the constant C independent of ε.*

Similarly to (3.50), but using the discrete operators (4.7), (4.59) and (4.29), we reformulate the equation (4.43) as

$$U^\varepsilon = \mathcal{P}^\varepsilon \hat{F}^\varepsilon(U^\varepsilon) + S^\varepsilon u_0^\varepsilon + I_0^\varepsilon \;, \tag{4.60}$$

where $\mathcal{P}^\varepsilon \stackrel{\text{def}}{=} \mathcal{K}_{\bar{\gamma}}^\varepsilon + R_\gamma^\varepsilon \mathcal{R}^\varepsilon$ and U^ε is a discrete modelled distribution.

Remark 4.3.7. If Z^ε is a canonical discrete model, then it follows from (4.40), (4.59), (4.7), Definition 4.2.1 and Assumption 4.3.6 that

$$u_t^\varepsilon(x) = \big(\mathcal{R}_t^\varepsilon U_t^\varepsilon\big)(x) \;, \qquad (t, x) \in \mathbb{R}_+ \times \Lambda_\varepsilon^d \;. \tag{4.61}$$

is a solution of the equation (4.43).

The following result can be proven in the same way as Theorem 3.3.11.

Theorem 4.3.8. *Let Z^ε be a sequence of models and let u_0^ε be a sequence of periodic functions on Λ_ε^d. Let furthermore the assumptions of Theorem 3.3.11 and Assumption 4.3.6 be satisfied. Then there exists $T_\star \in (0, +\infty]$ such that for every $T < T_\star$ the sequence of solution maps $\mathcal{S}_T^\varepsilon : (u_0^\varepsilon, Z^\varepsilon) \mapsto U^\varepsilon$ of the equation* (4.60) *is jointly Lipschitz continuous (uniformly in ε!) in the sense of Theorem 3.3.11, but for the discrete objects.*

Remark 4.3.9. Since we require uniformity in ε in Theorem 4.3.8, the solution of equation (4.60) is considered only up to some time point $T_\star$.

4.4 Inhomogeneous Gaussian models

In this section we analyse discrete and continuous models which are built from Gaussian noises. In the discrete case, we will work as usual on the grid Λ_ε^d, with $\varepsilon = 2^{-N}$ and $N \in \mathbb{N}$, and with the time-space scaling $\mathfrak{s} = (\mathfrak{s}_0, 1, \dots, 1)$.

We assume that we are given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, together with a white noise ξ over the Hilbert space $H \stackrel{\text{def}}{=} L^2(D)$ (see [Nua06]), where $D \stackrel{\text{def}}{=} \mathbb{R} \times \mathbb{T}^d$ and $\mathbb{T} \stackrel{\text{def}}{=} \mathbb{R}/\mathbb{Z}$ is the unit circle. In the sequel, we will always identify ξ with its periodic extension to $\mathbb{R}^{d+1}$.

In order to build a spatial discretisation of ξ, we take a compactly supported function $\varrho : \mathbb{R}^d \to \mathbb{R}$, such that for every $y \in \mathbb{Z}^d$ one has

$$\int_{\mathbb{R}^d} \varrho(x)\varrho(x-y)\,dx = \delta_{0,y}\;,$$

where $\delta_{\cdot,\cdot}$ is the Kronecker's function. Then, for $x \in \Lambda_\varepsilon^d$, we define the scaled function $\varrho_x^\varepsilon(y) \stackrel{\text{def}}{=} \varepsilon^{-d}\varrho((y-x)/\varepsilon)$ and

$$\xi^\varepsilon(t,x) \stackrel{\text{def}}{=} \xi(t, \varrho_x^\varepsilon)\;, \qquad (t,x) \in \mathbb{R} \times \Lambda_\varepsilon^d\;. \tag{4.62}$$

One can see that ξ^ε is a white noise on the Hilbert space $H_\varepsilon \stackrel{\text{def}}{=} L^2(\mathbb{R}) \otimes \ell^2(\mathbb{T}_\varepsilon^d)$, where $\mathbb{T}_\varepsilon \stackrel{\text{def}}{=} (\varepsilon\mathbb{Z})/\mathbb{Z}$ and $\ell^2(\mathbb{T}_\varepsilon^d)$ is equipped with the inner product $\langle \cdot, \cdot \rangle_\varepsilon$, defined in (4.2).

In the setting of Section 3.3.2, we assume that $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ is a discrete model on $\hat{\mathcal{T}}$ such that, for each $\tau \in \hat{\mathcal{F}}$ and each test function φ, the maps $\langle \Pi_x^{\varepsilon,t}\tau, \varphi \rangle_\varepsilon$,

$\Gamma^{\varepsilon,t}_{xy}\tau$ and $\Sigma^{\varepsilon,st}_{x}\tau$ belong to the inhomogeneous Wiener chaos of order $\|\tau\|$ (the number of occurrences of Ξ in τ) with respect to ξ^ε. Moreover, we assume that the distributions of the functions $(t,x) \mapsto \langle \Pi^{\varepsilon,t}_x \tau, \varphi_x\rangle_\varepsilon$, $(t,x) \mapsto \Gamma^{\varepsilon,t}_{x,x+h_1}\tau$ and $(t,x) \mapsto \Sigma^{\varepsilon,t,t+h_2}_x \tau$ are stationary, for all $h_1 \in \Lambda^d_\varepsilon$ and $h_2 \in \mathbb{R}$. In what follows, we will call the discrete models with these properties *stationary Gaussian discrete models*.

The following result provides a criterion for such a model to be bounded uniformly in ε. In its statement we use the following set:

$$\hat{\mathcal{F}}^- \stackrel{\text{def}}{=} \Big(\{\tau \in \hat{\mathcal{F}} : |\tau| < 0\} \cup \mathcal{F}^{\text{gen}}\Big) \setminus \mathcal{F}_{\text{poly}} \,. \tag{4.63}$$

Theorem 4.4.1. *In the described context, let $\hat{\mathscr{T}} = (\hat{\mathcal{T}}, \mathcal{G})$ be a truncated regularity structure and let $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ be an admissible stationary Gaussian discrete model on it. Let furthermore the bounds*

$$\mathbb{E}\Big[\|\Gamma^\varepsilon\|^{(\varepsilon)}_{\gamma;T}\Big]^p \lesssim 1\,, \qquad \mathbb{E}\Big[\|\Sigma^\varepsilon\|^{(\varepsilon)}_{\gamma;T}\Big]^p \lesssim 1\,. \tag{4.64}$$

hold uniformly in ε (see Remark 4.2.7) on the respective generating regularity structure $\mathscr{T}^{\text{gen}} = (\mathcal{T}^{\text{gen}}, \mathcal{G})$, for every $p \geq 1$, for every $\gamma > 0$ and for some $T \geq c$, where $c > 0$ is from Definition 4.2.10 and where the proportionality constants can depend on p. Let finally Π^ε be such that for some $\delta > 0$ and for each $\tau \in \hat{\mathcal{F}}^-$ the bounds

$$\begin{aligned} \mathbb{E}\Big[|\langle \Pi^{\varepsilon,t}_x \tau, \varphi^\lambda_x\rangle_\varepsilon|^2\Big] &\lesssim \lambda^{2|\tau|+\kappa}\,,\\ \mathbb{E}\Big[|\langle \big(\Pi^{\varepsilon,t}_x - \Pi^{\varepsilon,s}_x\big)\tau, \varphi^\lambda_x\rangle_\varepsilon|^2\Big] &\lesssim \lambda^{2(|\tau|-\delta)+\kappa}|t-s|^{2\delta/\mathfrak{s}_0}\,, \end{aligned} \tag{4.65}$$

hold uniformly in ε, all $\lambda \in [\varepsilon, 1]$, all $x \in \Lambda^d_\varepsilon$, all $s \neq t \in [-T, T]$ and all $\varphi \in \mathcal{B}^r_0(\mathbb{R}^d)$ with $r > -\lfloor \min \hat{\mathcal{A}} \rfloor$. Then, for every $\gamma > 0$, $p \geq 1$ and $\bar\delta \in [0, \delta)$, one has the following bound on $\hat{\mathscr{T}}$ uniformly in ε:

$$\mathbb{E}\Big[\|Z^\varepsilon\|^{(\varepsilon)}_{\bar\delta,\gamma;T}\Big]^p \lesssim 1\,. \tag{4.66}$$

Finally, let $\bar Z^\varepsilon = (\bar\Pi^\varepsilon, \bar\Gamma^\varepsilon, \bar\Sigma^\varepsilon)$ be another admissible stationary Gaussian discrete model on $\hat{\mathcal{T}}$, such that for some $\theta > 0$ and some $\bar\varepsilon > 0$ the maps $\Gamma^\varepsilon - \bar\Gamma^\varepsilon$, $\Sigma^\varepsilon - \bar\Sigma^\varepsilon$ and $\Pi^\varepsilon - \bar\Pi^\varepsilon$ satisfy the bounds (4.64) *and* (4.65) *respectively with* *proportionality constants of order $\bar{\varepsilon}^{2\theta}$. Then, for every $\gamma > 0$, $p \geq 1$ and $\bar{\delta} \in [0, \delta)$, the models Z^ε and $\bar{Z}^\varepsilon$ satisfy on $\hat{\mathscr{T}}$ the following bound uniformly in ε:*

$$\mathbb{E}\Big[\|\!| Z^\varepsilon ; \bar{Z}^\varepsilon \|\!|^{(\varepsilon)}_{\bar{\delta}, \gamma; T} \Big]^p \lesssim \bar{\varepsilon}^{\theta p} \; . \tag{4.67}$$

Proof. Since by assumption $\langle \Pi_x^{\varepsilon, t} \tau, \varphi \rangle_\varepsilon$ belongs to a fixed inhomogeneous Wiener chaos, the equivalence of moments [Nel73] and the bounds (4.65) yield the respective bounds on the p-th moments, for any $p \geq 1$. In particular, the Kolmogorov continuity criterion implies that for such p the bounds

$$\begin{aligned} \mathbb{E}\Big[\sup_{t \in [-T, T]} |\langle \Pi_x^{\varepsilon, t} \tau, \varphi_x^\lambda \rangle_\varepsilon| \Big]^p &\lesssim \lambda^{p|\tau| + \bar{\kappa}} \; , \\ \mathbb{E}\Big[\sup_{s \neq t \in [-T, T]} \frac{|\langle \big(\Pi_x^{\varepsilon, t} - \Pi_x^{\varepsilon, s}\big) \tau, \varphi_x^\lambda \rangle_\varepsilon|}{|t - s|^{\delta / \mathfrak{s}_0}} \Big]^p &\lesssim \lambda^{p(|\tau| - \delta) + \bar{\kappa}} \; , \end{aligned} \tag{4.68}$$

hold uniformly over x, φ and λ as in (4.65) and for some $\bar{\kappa} > 0$ depending on p. Going now by induction from the elements of $\mathcal{T}^{\mathrm{gen}}$ to the elements of $\hat{\mathcal{T}}$, using Lemmas 4.2.12 and 4.2.13 and the discrete multiresolution analysis defined in Section 4.2.1.2, we can obtain (4.66) in the same way as in the proof of [Hai14, Thm. 10.7]. The bound (4.67) can be proved similarly. $\square$

The conditions (4.65) can be checked quite easily if the maps $\Pi^\varepsilon \tau$ have certain Wiener chaos expansions. More precisely, we assume that there exist kernels $\mathcal{W}^{(\varepsilon; k)} \tau$ such that $\big(\mathcal{W}^{(\varepsilon; k)} \tau\big)(z) \in H_\varepsilon^{\otimes k}$, for $z \in \mathbb{R} \times \Lambda_\varepsilon^d$, and

$$\langle \Pi_0^{\varepsilon, t} \tau, \varphi \rangle_\varepsilon = \sum_{k \leq \|\tau\|} I_k^\varepsilon \Big(\int_{\Lambda_\varepsilon^d} \varphi(y) \, \big(\mathcal{W}^{(\varepsilon; k)} \tau\big)(t, y) \, dy \Big) \; , \tag{4.69}$$

where I_k^ε is the k-th order Wiener integral with respect to ξ^ε and the space H_ε is introduced above. Then we define the function

$$\big(\mathcal{K}^{(\varepsilon; k)} \tau\big)(z_1, z_2) \stackrel{\mathrm{def}}{=} \langle \big(\mathcal{W}^{(\varepsilon; k)} \tau\big)(z_1), \big(\mathcal{W}^{(\varepsilon; k)} \tau\big)(z_2) \rangle_{H_\varepsilon^{\otimes k}} \; , \tag{4.70}$$

for $z_1 \neq z_2 \in \mathbb{R} \times \Lambda_\varepsilon^d$, assuming that the expression on the right-hand side is well-defined.

In the same way, we assume that the maps $\bar{\Pi}^\varepsilon \tau$ are given by (4.69) via the respective kernels $\bar{\mathcal{W}}^{(\varepsilon; k)} \tau$. Moreover, we define the functions $\delta \mathcal{K}^{(\varepsilon; k)} \tau$ as in (4.70), but via the kernels $\bar{\mathcal{W}}^{(\varepsilon;k)}\tau - \mathcal{W}^{(\varepsilon;k)}\tau$, and we assume that the functions $\mathcal{K}^{(\varepsilon;k)}\tau$ and $\delta\mathcal{K}^{(\varepsilon;k)}\tau$ depend on the time variables t_1 and t_2 only via $t_1 - t_2$, i.e.

$$\big(\mathcal{K}^{(\varepsilon;k)}\tau\big)_{t_1-t_2}(x_1,x_2) \stackrel{\text{def}}{=} \big(\mathcal{K}^{(\varepsilon;k)}\tau\big)(z_1,z_2)\;, \tag{4.71}$$

where $z_i = (t_i, x_i)$, and similarly for $\delta\mathcal{K}^{(\varepsilon;k)}\tau$.

The following result shows that the bounds (4.65) follow from corresponding bounds on these functions.

Proposition 4.4.2. *In the described context, we assume that for some $\tau \in \hat{\mathcal{F}}^-$ there are values $\alpha > |\tau| \vee (-d/2)$ and $\delta \in (0, \alpha + d/2)$ such that the bounds*

$$\begin{aligned} |\big(\mathcal{K}^{(\varepsilon;k)}\tau\big)_0(x_1,x_2)| &\lesssim \sum_{\zeta\geq 0} \big(\|0,x_1\|_{\mathfrak{s},\varepsilon} + \|0,x_2\|_{\mathfrak{s},\varepsilon}\big)^\zeta \|0,x_1-x_2\|_{\mathfrak{s},\varepsilon}^{2\alpha-\zeta}\;,\\ \frac{|\delta^{0,t}\big(\mathcal{K}^{(\varepsilon;k)}\tau\big)(x_1,x_2)|}{|t|^{2\delta/\mathfrak{s}_0}} &\lesssim \sum_{\zeta\geq 0} \big(\|t,x_1\|_{\mathfrak{s},\varepsilon} + \|t,x_2\|_{\mathfrak{s},\varepsilon}\big)^\zeta \|0,x_1-x_2\|_{\mathfrak{s},\varepsilon}^{2\alpha-2\delta-\zeta}\;,\end{aligned} \tag{4.72}$$

hold uniformly in ε for $t \in \mathbb{R}$, $x_1, x_2 \in \Lambda_\varepsilon^d$ and $k \leq \vert\!\vert\!\vert \tau \vert\!\vert\!\vert$, where the operator $\delta^{0,t}$ is defined in (3.4)*, where $\|z\|_{\mathfrak{s},\varepsilon} \stackrel{def}{=} \|z\|_{\mathfrak{s}} \vee \varepsilon$, and where the sums run over finitely many values of $\zeta \in [0, 2\alpha - 2\delta + d)$. Then the bounds* (4.65) *hold for τ with a sufficiently small value of $\kappa > 0$.*

Let furthermore (4.72) *hold for the function $\delta\mathcal{K}^{(\varepsilon;k)}\tau$ with the proportionality constant of order $\bar{\varepsilon}^{2\theta}$, for some $\theta > 0$. Then the required bounds on $\big(\Pi^\varepsilon - \bar{\Pi}^\varepsilon\big)\tau$ in Theorem 4.4.1 hold.*

Proof. We note that due to our assumptions on stationarity of the models, it is sufficient to check the conditions (4.65) only for $\langle \Pi_0^{\varepsilon,t}\tau, \varphi_0^\lambda\rangle_\varepsilon$ and $\langle \big(\Pi_0^{\varepsilon,t} - \Pi_0^{\varepsilon,0}\big)\tau, \varphi_0^\lambda\rangle_\varepsilon$, and respectively for the map $\bar{\Pi}^\varepsilon$.

We start with the proof of the first statement of this proposition. We denote by $\Pi_0^{(\varepsilon,k),t}\tau$ the component of $\Pi_0^{\varepsilon,t}\tau$ belonging to the k-th homogeneous Wiener chaos. Furthermore, we will use the following property of the Wiener integral [Nua06]:

$$\mathbb{E}\big[I_k^\varepsilon(f)^2\big] \leq \|f\|_{H_\varepsilon^{\otimes k}}\;, \qquad f \in H_\varepsilon^{\otimes k}\;. \tag{4.73}$$

Thus, from this property, (4.71) and the first bound in (4.72), we get

$$\mathbb{E}|\langle \Pi_0^{(\varepsilon,k),t}\tau, \varphi_0^\lambda\rangle_\varepsilon|^2 \lesssim \int_{\Lambda_\varepsilon^d}\int_{\Lambda_\varepsilon^d} |\varphi_0^\lambda(x_1)|\,|\varphi_0^\lambda(x_2)|\,|\big(\mathcal{K}^{(\varepsilon,k)}\tau\big)_0(x_1,x_2)|\,dx_1 dx_2$$

$$
\begin{aligned}
&\lesssim \lambda^{-2d} \sum_{\zeta \geq 0} \int_{\substack{|x_1| \leq \lambda \\ |x_2| \leq \lambda}} \big(\|0, x_1\|_{\mathfrak{s},\varepsilon} + \|0, x_2\|_{\mathfrak{s},\varepsilon} \big)^{\zeta} \|0, x_1 - x_2\|_{\mathfrak{s},\varepsilon}^{2\alpha - \zeta}\, dx_1 dx_2 \\
&\lesssim \lambda^{-2d} \sum_{\zeta \geq 0} \lambda^{d+\zeta} \int_{|x| \leq 2\lambda} \|0, x\|_{\mathfrak{s},\varepsilon}^{2\alpha - \zeta}\, dx \lesssim \lambda^{2\alpha}\;, \qquad (4.74)
\end{aligned}
$$

for $\lambda \geq \varepsilon$. Here, to have the proportionality constant independent of ε, we need $2\alpha - \zeta > -d$. Combining the bounds (4.74) for each k with stationarity of $\Pi^{\varepsilon}\tau$, we obtain the first estimate in (4.65), with a sufficiently small $\kappa > 0$.

Now, we will investigate the time regularity of the map Π^{ε}. For $|t| \geq \lambda^{\mathfrak{s}_0}$ we can use (4.74) and brutally bound

$$
\begin{aligned}
\mathbb{E}|\langle \delta^{0,t} \Pi_0^{(\varepsilon,k)} \tau, \varphi_0^{\lambda} \rangle_{\varepsilon}|^2 &\lesssim \mathbb{E}|\langle \Pi_0^{(\varepsilon,k),t} \tau, \varphi_0^{\lambda} \rangle_{\varepsilon}|^2 + \mathbb{E}|\langle \Pi_0^{(\varepsilon,k),0} \tau, \varphi_0^{\lambda} \rangle_{\varepsilon}|^2 \\
&\lesssim \lambda^{2\alpha} \lesssim |t|^{2\delta/s_0} \lambda^{2\alpha - 2\delta}\;, \qquad (4.75)
\end{aligned}
$$

for any $\delta \geq 0$, which is the required estimate. In the case $|t| < \lambda^{\mathfrak{s}_0}$, the bound (4.73) and second bound in (4.72) yield

$$
\begin{aligned}
\mathbb{E}|\langle \delta^{0,t} \Pi_0^{(\varepsilon,k)} \tau, \varphi_0^{\lambda} \rangle_{\varepsilon}|^2 &\lesssim \int_{\Lambda_\varepsilon^d} \int_{\Lambda_\varepsilon^d} |\varphi_0^{\lambda}(x_1)|\, |\varphi_0^{\lambda}(x_2)|\, |\delta^{0,t} \big(\mathcal{K}^{(\varepsilon,k)} \tau\big)(x_1, x_2)|\, dx_1 dx_2 \\
&\quad + \int_{\Lambda_\varepsilon^d} \int_{\Lambda_\varepsilon^d} |\varphi_0^{\lambda}(x_1)|\, |\varphi_0^{\lambda}(x_2)|\, |\delta^{-t,0} \big(\mathcal{K}^{(\varepsilon,k)} \tau\big)(x_1, x_2)|\, dx_1 dx_2 \\
&\lesssim |t|^{2\delta/\mathfrak{s}_0} \lambda^{-2d} \sum_{\zeta \geq 0} \int_{\substack{|x_1| \leq \lambda \\ |x_2| \leq \lambda}} \big(\|t, x_1\|_{\mathfrak{s},\varepsilon} + \|t, x_2\|_{\mathfrak{s},\varepsilon} \big)^{\zeta} \|0, x_1 - x_2\|_{\mathfrak{s},\varepsilon}^{2\alpha - 2\delta - \zeta} dx_1 dx_2 \\
&\lesssim |t|^{2\delta/\mathfrak{s}_0} \lambda^{2\alpha - 2\delta}\;, \qquad (4.76)
\end{aligned}
$$

where the integral is bounded as before for $2\alpha - 2\delta - \zeta > -d$. Combining the bounds (4.75) and (4.76) for each value of k with stationarity of $\Pi^{\varepsilon}\tau$, we obtain the second estimate in (4.65). The required bounds on $\big(\Pi^{\varepsilon} - \bar{\Pi}^{\varepsilon}\big)\tau$ can be proved in a similar way. $\square$

Remark 4.4.3. Let us be given an admissible continuous model $Z = (\Pi, \Gamma, \Sigma)$ on $\hat{\mathscr{T}}$ such that the map Π is given on $\hat{\mathcal{F}}^-$ by the expansions (4.69) in which we replace all the discrete objects by their continuous counterparts. Then one can prove in the same way the analogues of Theorem 4.4.1 and Proposition 4.4.2 in the continuous case, i.e. when we use $\varepsilon = 0$ and use continuous objects in place of the discrete ones.

4.4.1 Continuous inhomogeneous models

In this section we will show how in some cases we can build a continuous inhomogeneous model from an admissible model in the sense of [Hai14, Def. 8.29].

For a white noise ξ on a Hilbert space H as in the beginning of the previous section, we assume that we are given an admissible model $\tilde{Z} = (\tilde{\Pi}, \tilde{\Gamma})$ in the sense of [Hai14, Def. 8.29] on the truncated regularity structure $\hat{\mathscr{T}}$ such that for every $\tau \in \hat{\mathcal{F}}$, every test function φ on $\mathbb{R}^{d+1}$ and every pair of points $z, \bar{z} \in \mathbb{R}^{d+1}$, the maps $\langle \tilde{\Pi}_z \tau, \varphi \rangle$ and $\tilde{\Gamma}_{z\bar{z}} \tau$ belong to the inhomogeneous Wiener chaos of order $\vert\!\vert\!\vert \tau \vert\!\vert\!\vert$ (the quantity $\vert\!\vert\!\vert \tau \vert\!\vert\!\vert$ is defined in the beginning of Section 4.4) with respect to ξ. Furthermore, we assume that for every $\tau \in \hat{\mathcal{F}}$ there exist kernels $\mathcal{W}^{(k)}\tau$ such that for every test function φ on $\mathbb{R}^{d+1}$ one has $\int_{\mathbb{R}^{d+1}} \varphi(z) \big(\mathcal{W}^{(k)}\tau\big)(z)\, dz \in H^{\otimes k}$, postulating that the integral is well-defined, and $\tilde{\Pi}_z \tau$ can be written as

$$\langle \tilde{\Pi}_z \tau, \varphi_z \rangle = \sum_{k \leq \vert\!\vert\!\vert \tau \vert\!\vert\!\vert} I_k \Big(S_z^{\otimes k} \int_{\mathbb{R}^{d+1}} \varphi(\bar{z}) \, \big(\mathcal{W}^{(k)}\tau\big)(\bar{z})\, d\bar{z} \Big) \;, \tag{4.77}$$

where I_k is the k-th Wiener integral with respect to ξ, φ_z is the recentered version of φ and $\{S_z\}_{z \in \mathbb{R}^{d+1}}$ is the group of translations acting on H. Using the scalar product in $H^{\otimes k}$ rather than in $H_\varepsilon^{\otimes k}$ and points from $\mathbb{R}^{d+1}$, we assume that the respective modification of the right-hand side of (4.70) is well defined and we introduce for these kernels the functions $\mathcal{K}^{(k)}\tau$. In addition, we assume that they satisfy the continuous analogue of (4.71) and the first bound in (4.72) (when $\varepsilon = 0$). Then for every $\tau \in \hat{\mathcal{F}}$ we can define a distribution $\Pi_x^t \tau \in \mathcal{S}'(\mathbb{R}^d)$ by

$$\langle \Pi_x^t \tau, \varphi_x \rangle = \sum_{k \leq \vert\!\vert\!\vert \tau \vert\!\vert\!\vert} I_k \Big(S_{(t,x)}^{\otimes k} \int_{\mathbb{R}^d} \varphi(y) \, \big(\mathcal{W}^{(k)}\tau\big)(t,y)\, dy \Big) \;, \tag{4.78}$$

where φ is a test function on $\mathbb{R}^d$. In fact, the expression on the right-hand side of (4.78) is well-defined, because one can show in exactly the same way as in (4.74) that for every test function φ on $\mathbb{R}^d$ one has

$$\Big| \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \varphi_0^\lambda(x_1)\, \varphi_0^\lambda(x_2) \, \big(\mathcal{K}^{(k)}\tau\big)_0(x_1, x_2)\, dx_1 dx_2 \Big| \lesssim \lambda^{2\alpha} \;.$$

Finally, defining the maps Γ and Σ by

$$\Gamma^t_{xy} = \tilde{\Gamma}_{(t,x),(t,y)} , \qquad \Sigma^{st}_x = \tilde{\Gamma}_{(s,x),(t,x)} , \tag{4.79}$$

one can see that (Π, Γ, Σ) is an admissible inhomogeneous model on $\hat{\mathscr{T}}$.

Chapter 5

Convergence of the discrete dynamical Φ^4_3 model

5.1 Introduction and statements of the results

In this chapter we consider discretisations of a particular rough stochastic PDE, the dynamical Φ^4 model in dimension 3, which can be formally described by the equation (3.2). It is natural to consider finite difference approximations to (3.2) for a number of reasons. Our main motivation goes back to the seminal article [BFS83], where the authors provide a very clean and relatively compact argument showing that lattice approximations μ_ε to the Φ^4_3 measure are tight as the mesh size goes to 0. These measures are given on the dyadic grid $\mathbb{T}^3_\varepsilon \subset \mathbb{T}^3$ with the mesh size $\varepsilon > 0$ by

$$\mu_\varepsilon(\Phi^\varepsilon) \stackrel{\text{def}}{=} e^{-S_\varepsilon(\Phi^\varepsilon)} \prod_{x \in \mathbb{T}^3_\varepsilon} d\Phi^\varepsilon(x)/Z_\varepsilon \;,$$

for every function Φ^ε on $\mathbb{T}^3_\varepsilon$, where Z_ε is a normalisation factor, called "partition function", and the "action" S_ε is defined by

$$S_\varepsilon(\Phi^\varepsilon) \stackrel{\text{def}}{=} \varepsilon \sum_{x \sim y} \big(\Phi^\varepsilon(x) - \Phi^\varepsilon(y)\big)^2 - \frac{C^{(\varepsilon)}\varepsilon^3}{2} \sum_{x \in \mathbb{T}^3_\varepsilon} \Phi^\varepsilon(x)^2 + \frac{\varepsilon^3}{4} \sum_{x \in \mathbb{T}^3_\varepsilon} \Phi^\varepsilon(x)^4 \;, \tag{5.1}$$

with $C^{(\varepsilon)}$ being a "renormalisation constant" and with the first sum running over all the nearest neighbours on the grid $\mathbb{T}^3_\varepsilon$, when each pair x, y is counted twice. Since these measures are invariant for the natural finite difference approximation of (3.2), showing that these converge to (3.2) straightforwardly implies that any accumulation point of μ_ε is invariant for the solutions of (3.2). These accumulation points are known to coincide with the Φ^4_3 measure μ [Par77], thus showing that μ is indeed invariant for (3.2), as one might expect. Heuristically, the measure μ can be written as

$$\mu(\Phi) \sim e^{-S(\Phi)} \prod_{x \in \mathbb{T}^3} d\Phi(x) \;, \tag{5.2}$$

for every $\Phi \in \mathcal{S}'$. In this case the "action" S as a limit of its finite difference approximations (5.1), i.e. it is formally given by

$$S(\Phi) = \int_{\mathbb{T}^3} \left(\frac{1}{2}\big(\nabla\Phi(x)\big)^2 - \frac{\infty}{2}\Phi(x)^2 + \frac{1}{4}\Phi(x)^4 \right) dx \;.$$

Another reason why discretisations of (3.2) are interesting is because they can be related to the behaviour of Ising-type models under Glauber dynamics near their critical temperature, see [SG73, GRS75]. See also the related result [MW14] where the dynamical Φ^4_2 model is obtained from the Glauber dynamic for a Kac-Ising model in a more direct way, without going through lattice approximations. Similar results are expected to hold in three spatial dimensions, see e.g. the review article [GLP99].

We will henceforth consider discretisations of (3.2) of the form

$$\frac{d}{dt}\Phi^\varepsilon = \Delta^\varepsilon \Phi^\varepsilon + C^{(\varepsilon)}\Phi^\varepsilon - \big(\Phi^\varepsilon\big)^3 + \xi^\varepsilon\;, \qquad \Phi^\varepsilon(0,\cdot) = \Phi^\varepsilon_0(\cdot)\;, \tag{5.3}$$

on the dyadic discretisation $\mathbb{T}^3_\varepsilon$ of $\mathbb{T}^3$ with mesh size $\varepsilon = 2^{-N}$ for $N \in \mathbb{N}$, where $\Phi^\varepsilon_0 \in \mathbb{R}^{\mathbb{T}^3_\varepsilon}$, Δ^ε is the nearest-neighbor approximation of the Laplacian Δ, and ξ^ε is a spatial discretisation of ξ. We construct these discretisations on a common probability space by setting

$$\xi^\varepsilon(t,x) \stackrel{\text{def}}{=} \varepsilon^{-3}\langle \xi(t,\cdot), \mathbf{1}_{|\cdot - x|_\infty \le \varepsilon/2}\rangle\;, \qquad (t,x) \in \mathbb{R}\times\mathbb{T}^3_\varepsilon\;, \tag{5.4}$$

where $|x|_\infty$ denotes the supremum norm of $x \in \mathbb{R}^3$. Our results are however flexible enough to easily accommodate a variety of different approximations to the noise and the Laplacian.

Existence and uniqueness of global solutions to (5.3) for any fixed $\varepsilon > 0$ follows immediately from standard results for SDEs [Kha12, IW89]. Our main approximation result is the following, where we take the initial conditions Φ^ε_0 to be random variables defined on a common probability space, independent of the noise ξ. We could of course simply take them deterministic, but this formulation will be how it will then be used in our proof of existence of global solutions.

Theorem 5.1.1. *Let ξ be a space-time white noise over $L^2(\mathbb{R}\times\mathbb{T}^3)$ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, let $\Phi_0 \in \mathcal{C}^\eta(\mathbb{R}^3)$ almost surely, for some $\eta > -\frac{2}{3}$, and let Φ be the unique maximal solution of* (3.2) *on $[0, T_\star)$. Let furthermore Δ^ε be the nearest-neighbor approximation of Δ, let $\Phi^\varepsilon_0 \in \mathbb{R}^{\mathbb{T}^3_\varepsilon}$ be a random variable on the same probability space, let ξ^ε be given by* (5.4)*, and let Φ^ε be the unique global solution of* (5.3). *If the initial data satisfy almost surely*

$$\lim_{\varepsilon\to 0} \|\Phi_0; \Phi^\varepsilon_0\|^{(\varepsilon)}_{\mathcal{C}^\eta} = 0\;,$$

then for every $\alpha < -\frac{1}{2}$ there is a sequence of renormalisation constants $C^{(\varepsilon)} \sim \varepsilon^{-1}$ in (5.3) *and a sequence of stopping times T_ε satisfying $\lim_{\varepsilon\to 0} T_\varepsilon = T_\star$ in probability* *such that, for every $\bar{\eta} < \eta \wedge \alpha$, and for any $\delta > 0$ small enough, one has the limit in probability*

$$\lim_{\varepsilon \to 0} \|\Phi; \Phi^\varepsilon\|^{(\varepsilon)}_{\mathcal{C}^{\delta,\alpha}_{\bar{\eta},T_\varepsilon}} = 0\ . \tag{5.5}$$

As a corollary of this convergence result and an argument along the lines of [Bou94], we have the following result, where we denote by μ the Φ^4_3 measure on the torus, heuristically given in (5.2).

Corollary 5.1.2. *For μ-almost every initial condition Φ_0 and for every $T > 0$, the solution of* (3.2) *constructed in [Hai14] belongs to $\mathcal{C}^{\delta,\alpha}_{\bar{\eta}}\big([0,T], \mathbb{T}^3\big)$, for δ, α and $\bar{\eta}$ as in* (5.5)*. In particular, this yields a reversible Markov process on $\mathcal{C}^\alpha\big(\mathbb{T}^3\big)$ which admits μ as an invariant measure.*

In order to prove this result, we will use regularity structures and discrete models for them developed in Chapters 3 and 4.

Recently the convergence of solutions of (5.3) to those of (3.2) has been obtained by Zhu and Zhu [ZZ15a] using different methods. Additionally, Gubinelli and Perkowski [GP15] obtained a similar result also for the KPZ equation. One advantage of the approach pursued here is that it is quite systematic and that many of our intermediate results do not specifically refer to the Φ^4_3 model.

Structure of the chapter

In Section 5.2, we rewrite the solution of (5.3) using the regularity structures and discrete models introduced in the previous chapters. Section 5.3 is devoted to analysis of singular functions, from which convergence of the discrete models follows. Finally, in Section 5.4 we prove Theorem 5.1.1 and Corollary 5.1.2. Throughout this chapter we use the notations of Chapters 3 and 4.

5.2 Reformulation of the discrete dynamical Φ^4_3 model

In this section we use the theory developed above to prove convergence of the solutions of (5.3), where Δ^ε is the nearest-neighbor approximation of Δ and the discrete noise ξ^ε is defined in (5.4) via a space-time white noise ξ.

Example 4.3.2 yields that Assumption 4.3.1 is satisfied, and moreover ξ^ε is a discrete noise as in (4.62). The time-space scaling for the equation (3.2) is

$\mathfrak{s} = (2,1,1,1)$ and the kernels K and K^ε are defined in Lemma 4.3.4 with the parameters $\beta = 2$ and $r > 2$, for the operators Δ and Δ^ε respectively.

The regularity structure $\mathscr{T} = (\mathcal{T}, \mathcal{G})$ for the equation (3.2), introduced in Section 3.3.1, has the model space $\mathcal{T} = \mathrm{span}\{\mathcal{F}\}$, where

$$\begin{aligned}\mathcal{F} = \{\mathbf{1}, \Xi, \Psi, \Psi^2, \Psi^3, \Psi^2 X_i, \mathcal{I}(\Psi^3)\Psi, \mathcal{I}(\Psi^3)\Psi^2, \mathcal{I}(\Psi^2)\Psi^2, \mathcal{I}(\Psi^2),& \\ \mathcal{I}(\Psi)\Psi, \mathcal{I}(\Psi)\Psi^2, X_i, \ldots\}&\;,\end{aligned} \tag{5.6}$$

$\Psi \stackrel{\text{def}}{=} \mathcal{I}(\Xi)$, $|\Xi| = \alpha \in \big(-\frac{18}{7}, -\frac{5}{2}\big)$ and the index i corresponds to any of the three spatial dimensions, see [Hai14, Sec. 9.2]. The homogeneities $\mathcal{A}$ of the symbols in $\mathcal{F}$ are defined recursively by the rules (3.44). The bound $\alpha > -\frac{18}{7}$ is required, in order to have a fixed regularity structure, otherwise we need to add extra symbols to the set $\hat{\mathcal{F}}^-$ defined in (5.8) below.

A two-parameter renormalisation subgroup $\mathfrak{R}^0 \subset \mathfrak{R}$ for this problem consists of the linear maps M on $\mathcal{T}$, defined by

$$\begin{aligned} M\Psi^2 &= \Psi^2 - C_1\mathbf{1}\;, \\ M\big(\Psi^2 X_i\big) &= \Psi^2 X_i - 3C_1 X_i\;, \\ M\Psi^3 &= \Psi^3 - C_1\Psi\;, \\ M\big(\mathcal{I}\big(\Psi^2\big)\Psi^2\big) &= \mathcal{I}\big(\Psi^2\big)\big(\Psi^2 - C_1\mathbf{1}\big) - C_2\mathbf{1}\;, \\ M\mathcal{I}\big(\Psi^3\big) &= \mathcal{I}\big(\Psi^3\big) - 3C_1\mathcal{I}\big(\Psi\big)\;, \\ M\big(\mathcal{I}\big(\Psi^3\big)\Psi\big) &= \big(\mathcal{I}\big(\Psi^3\big) - 3C_1\mathcal{I}\big(\Psi\big)\big)\Psi\;, \\ M\big(\mathcal{I}\big(\Psi^3\big)\Psi^2\big) &= \big(\mathcal{I}\big(\Psi^3\big) - 3C_1\mathcal{I}\big(\Psi\big)\big)\big(\Psi^2 - C_1\mathbf{1}\big) - 3C_2\Psi\;, \\ M\big(\mathcal{I}\big(\Psi\big)\Psi^2\big) &= \mathcal{I}\big(\Psi\big)\big(\Psi^2 - C_1\mathbf{1}\big)\;, \end{aligned} \tag{5.7}$$

which can be extended to the remaining elements in $\mathcal{F}$ (see [Hai14, Sec. 9.2]), and where C_1 and C_2 are the two parameter constants.

In the proof of Theorem 5.1.1 in Section 5.4 we will make use of the Gaussian models on $\mathscr{T}$ built in [Hai14, Thm. 10.22]. As one can see from Remark 4.4.3 and the continuous versions of the bounds (4.72), one can expect a concrete realisation of an abstract symbol τ to be a function in time if $|\tau| > -\frac{3}{2}$. In our case, the symbols Ξ and Ψ^3 don't satisfy this condition, having homogeneities $\alpha < -\frac{5}{2}$ and $3(\alpha + 2) < -\frac{3}{2}$ respectively. This was exactly the reason for introducing a truncated regularity structure in Section 3.3.2, which primarily means that we can remove these problematic symbols from $\mathscr{T}$. More precisely, we introduce a new symbol $\bar{\Psi} \stackrel{\text{def}}{=} \mathcal{I}(\Psi^3)$ and the set

$$\mathcal{F}^{\text{gen}} \stackrel{\text{def}}{=} \{\Psi, \bar{\Psi}\} \cup \mathcal{F}_{\text{poly}} \;.$$

Furthermore, we remove Ξ and Ψ^3 from $\mathcal{F}$ in (5.6) and replace all the occurrences of $\mathcal{I}(\Psi^3)$ by $\bar{\Psi}$, which gives

$$\hat{\mathcal{F}} = \{\mathbf{1}, \Psi, \Psi^2, \Psi^2 X_i, \Psi\bar{\Psi}, \Psi^2\bar{\Psi}, \mathcal{I}(\Psi^2)\Psi^2, \mathcal{I}(\Psi^2), \mathcal{I}(\Psi)\Psi, \mathcal{I}(\Psi)\Psi^2, X_i, \ldots\} \;.$$

Then the model spaces of the regularity structures $\mathscr{T}^{\text{gen}}$ and $\hat{\mathscr{T}}$ from Definition 3.3.5 are the linear spans of $\mathcal{F}^{\text{gen}}$ and $\hat{\mathcal{F}}$ respectively, and the set $\hat{\mathcal{F}}^-$ from (4.63) is given in this case by

$$\hat{\mathcal{F}}^- = \{\Psi,\ \bar{\Psi},\ \Psi^2,\ \Psi^2 X_i,\ \Psi\bar{\Psi},\ \mathcal{I}(\Psi^2)\Psi^2,\ \Psi^2\bar{\Psi}\} \;. \tag{5.8}$$

In the following lemma we show that the nonlinearities in (3.2) and (5.3) satisfy the required assumptions, provided that the appearance of the renormalisation constant is being dealt with at the level of the corresponding models.

Lemma 5.2.1. *Let $\hat{\alpha} \stackrel{\text{def}}{=} \min \hat{\mathcal{A}}$. Then, for any $\gamma > |2\hat{\alpha}|$ and any $\eta \leq \hat{\alpha}$, the maps*

$$F(\tau) = F^\varepsilon(\tau) = -\mathcal{Q}_{\leq 0}(\tau^3) + \Xi$$

satisfy Assumptions 3.3.9 and 4.3.6 with

$$F_0 = F_0^\varepsilon = \Xi - \Psi^3\;, \qquad I_0 = I_0^\varepsilon = \Psi - \bar{\Psi}\;,$$

and $\bar{\gamma} = \gamma + 2\hat{\alpha}$, $\bar{\eta} = 3\eta$.

Proof. The space $\mathcal{T}_{\mathcal{U}} \subset \hat{\mathcal{T}}$ introduced in Section 3.3.1 is spanned by polynomials and elements of the form $\mathcal{I}(\tau)$. Thus, the fact that the function $\hat{F}$ defined in (3.48) maps $\{I_0(z) + \tau \,:\, \tau \in \hat{\mathcal{T}} \cap \mathcal{T}_{\mathcal{U}}\}$ into $\hat{\mathcal{T}}$ is obvious. The bounds (3.51) in the continuous and discrete cases can be proved in exactly the same way as in [Hai14, Prop. 6.12], using Remarks 3.2.10 and 4.2.4 respectively. □

Our following aim is to define a discrete model $Z^\varepsilon = (\Pi^\varepsilon, \Gamma^\varepsilon, \Sigma^\varepsilon)$ on $\mathscr{T}^{\text{gen}}$, and to extend it in the canonical way to $\hat{\mathscr{T}}$ as in Remark 4.2.14. To this end, we postulate, for $s, t \in \mathbb{R}$ and $x, y \in \Lambda_\varepsilon^3$,

$$\big(\Pi_x^{\varepsilon,t}\Psi\big)(y) = \big(K^\varepsilon \star_\varepsilon \xi^\varepsilon\big)(t,y)\;, \qquad \Gamma_{xy}^{\varepsilon,t}\Psi = \Psi\;, \qquad \Sigma_x^{\varepsilon,st}\Psi = \Psi\;. \tag{5.9}$$

Furthermore, we denote the function $\bar\psi^\varepsilon(t,x) \stackrel{\text{def}}{=} \big(K^\varepsilon \star_\varepsilon \big(\Pi_x^{\varepsilon,t}\Psi\big)^3\big)(t,x)$ and set

$$\begin{aligned}\big(\Pi_x^{\varepsilon,t}\bar\Psi\big)(y) = \bar\psi^\varepsilon(t,y) - \bar\psi^\varepsilon(t,x)\;, \qquad \Gamma_{xy}^{\varepsilon,t}\bar\Psi = \bar\Psi - \big(\bar\psi^\varepsilon(t,y) - \bar\psi^\varepsilon(t,x)\big)\mathbf{1}\;,\\ \Sigma_x^{\varepsilon,st}\bar\Psi = \bar\Psi - \big(\bar\psi^\varepsilon(t,x) - \bar\psi^\varepsilon(s,x)\big)\mathbf{1}\;.\end{aligned} \tag{5.10}$$

Postulating the actions of these maps on the abstract polynomials in the standard way, we canonically extend Z^ε to the whole $\hat{\mathscr{T}}$.

Furthermore, we define the renormalisation constants[1]

$$C_1^{(\varepsilon)} \stackrel{\text{def}}{=} \int_{\mathbb{R}\times\Lambda_\varepsilon^3} \big(K^\varepsilon(z)\big)^2\, dz\;, \quad C_2^{(\varepsilon)} \stackrel{\text{def}}{=} 2\int_{\mathbb{R}\times\Lambda_\varepsilon^3} \big(K^\varepsilon \star_\varepsilon K^\varepsilon\big)(z)^2\, K^\varepsilon(z)\, dz\;, \tag{5.11}$$

and use them to define the renormalisation map M^ε as in (5.7). Finally, we define the renormalised model $\hat Z^\varepsilon$ for Z^ε and M^ε as in Remark 4.2.16. Using the model $\hat Z^\varepsilon$ in (4.61) we obtain a solution to the discretised Φ_3^4 eqaution (5.3) with

$$C^{(\varepsilon)} \stackrel{\text{def}}{=} 3C_1^{(\varepsilon)} - 9C_2^{(\varepsilon)}\;.$$

Before giving a proof of Theorem 5.1.1 we provide some technical results.

5.3 Functions with prescribed singularities

It follows from Proposition 4.4.2 and Remark 4.4.3 that the "strength" of singularity of a kernel determines the regularity of the respective distribution. In this section we provide some properties of singular continuous and discrete functions. As usual we fix a scaling $\mathfrak{s} = (\mathfrak{s}_0, 1, \ldots, 1)$ of $\mathbb{R}^{d+1}$ with $\mathfrak{s}_0 \geq 1$.

5.3.1 Continuous functions with singularities

In this section we list some properties of continuous functions with singularities at the origin. Proofs of all these results immediately follow from those of [Hai14,

[1] One can show that $C_1^{(\varepsilon)} \sim \varepsilon^{-1}$ and $C_2^{(\varepsilon)} \sim \log\varepsilon$.

Sec. 10.3].

Let $K : \mathbb{R}^{d+1} \setminus \{0\} \to \mathbb{R}$ be supported in a ball centered at the origin. We say that K is of order $\zeta \in \mathbb{R}$, if for some $m \geq 0$ one has

$$\llbracket K \rrbracket_{\zeta;m} \stackrel{\text{def}}{=} \max_{|k|_\mathfrak{s} \leq m} \sup_{z \neq 0} \frac{|D^k K(z)|}{\|z\|_\mathfrak{s}^{(\zeta - |k|_\mathfrak{s}) \wedge 0}} < \infty , \tag{5.12}$$

where $z \in \mathbb{R}^{d+1}$ and $k \in \mathbb{N}^{d+1}$. The following result establishes how product and convolution change orders of singularities.

Lemma 5.3.1. *Let functions K_1 and K_2 be of orders ζ_1 and ζ_2 respectively. Then we have the following results:*

- *If $\zeta_1, \zeta_2 \leq 0$, then $K_1 K_2$ is of order $\zeta_1 + \zeta_2$ and for every $m \geq 0$ one has*

$$\llbracket K_1 K_2 \rrbracket_{\zeta_1 + \zeta_2; m} \lesssim \llbracket K_1 \rrbracket_{\zeta_1;m} \llbracket K_2 \rrbracket_{\zeta_2;m} .$$

- *If $\zeta_1 \wedge \zeta_2 > -|\mathfrak{s}|$ and $\bar{\zeta} \stackrel{\text{def}}{=} \zeta_1 + \zeta_2 + |\mathfrak{s}| \notin \mathbb{N}$, then the function $K_1 \star K_2$ has all derivatives at the origin of orders k such that $|k|_\mathfrak{s} < \bar{\zeta}$ and satisfies the bound*

$$\llbracket K_1 \star K_2 \rrbracket_{\bar{\zeta};m} \lesssim \llbracket K_1 \rrbracket_{\zeta_1;m} \llbracket K_2 \rrbracket_{\zeta_2;m} .$$

In all these estimates the proportionality constants depend only on the support of the functions K_i.

Sometimes we need to bound a spatial increment of a singular function. The following lemma provides a relevant result.

Lemma 5.3.2. *Let a function K be of order $\zeta \leq 0$. Then for every $\kappa \in [0,1]$, $t \in \mathbb{R}$ and $x_1, x_2 \in \mathbb{R}^d$ one has*

$$|K(t,x_1) - K(t,x_2)| \lesssim |x_1 - x_2|^\kappa \Big(\|t, x_1\|_\mathfrak{s}^{\zeta - \kappa} + \|t, x_2\|_\mathfrak{s}^{\zeta-\kappa} \Big) \llbracket K \rrbracket_{\zeta;1} .$$

For a singular function K of order $\zeta \in (-|\mathfrak{s}| - 1, -|\mathfrak{s}|]$, we define the distribution $\mathscr{R}K$ by its action on the test functions φ on $\mathbb{R}^{d+1}$ in the following way:

$$(\mathscr{R}K)(\varphi) \stackrel{\text{def}}{=} \int_{\mathbb{R}^{d+1}} K(z)(\varphi(z) - \varphi(0))\, dz .$$

The following result provides an estimate on $\mathscr{R}K$.

Lemma 5.3.3. *Let functions K_1 and K_2 be of orders ζ_1 and ζ_2 respectively with $\zeta_1 \in \big(-|\mathfrak{s}| - 1, -|\mathfrak{s}|\big]$ and $\zeta_2 \in \big(-2|\mathfrak{s}| - \zeta_1, 0\big]$. Then the function $\big(\mathscr{R}K_1\big) \star K_2$ is of order $\bar{\zeta} \stackrel{\text{def}}{=} \zeta_1 + \zeta_2 + |\mathfrak{s}|$ and, for any $m \geq 0$, one has*

$$\llbracket (\mathscr{R}K_1) \star K_2 \rrbracket_{\bar{\zeta};m} \lesssim \llbracket K_1 \rrbracket_{\zeta_1;m} \llbracket K_2 \rrbracket_{\zeta_2;m+\mathfrak{s}_0} .$$

Now, we will show how convolutions with mollifiers change singular functions. For this we fix a smooth compactly supported function $\varrho : \mathbb{R}^{d+1} \to \mathbb{R}$ which integrates to 1, and we define its rescaled version $\varrho_0^{\varepsilon,\mathfrak{s}}$ as in (3.6). Then we have the following result.

Lemma 5.3.4. *In the described settings, let K be of order $\zeta \in (-|\mathfrak{s}|, 0)$. Then $K \star \varrho_0^{\varepsilon,\mathfrak{s}}$ is smooth and for every $\kappa \in [0,1]$ one has the bound*

$$\llbracket K - K \star \varrho_0^{\varepsilon,\mathfrak{s}} \rrbracket_{\zeta-\kappa;m} \lesssim \varepsilon^{\kappa} \llbracket K \rrbracket_{\zeta;m+\mathfrak{s}_0} .$$

5.3.2 Discrete functions with singularities

We provide below discrete analogues of the results from the previous section. All of them can be proved in a very similar way, and we give only some details at the most crucial points in the proofs.

For a function K^ε defined on $\mathbb{R} \times \Lambda_\varepsilon^d$ and supported in a ball centered at the origin, we denote by $D_{i,\varepsilon}$ the finite difference derivative, i.e.

$$D_{i,\varepsilon} K^\varepsilon(t,x) \stackrel{\text{def}}{=} \varepsilon^{-1}(K^\varepsilon(t, x + \varepsilon e_i) - K^\varepsilon(t,x)) ,$$

where $\{e_i\}_{i=1\ldots d}$ is the canonical basis of $\mathbb{R}^d$, and for $k = (k_0, k_1, \ldots, k_d) \in \mathbb{N}^{d+1}$ we define $D_\varepsilon^k \stackrel{\text{def}}{=} D_t^{k_0} D_{1,\varepsilon}^{k_1} \ldots D_{d,\varepsilon}^{k_d}$. We allow the function K^ε to be non-differentiable in time only on the set $P_0 \stackrel{\text{def}}{=} \{(0,x) : x \in \Lambda_\varepsilon^d\}$. Furthermore, we define for $\zeta \in \mathbb{R}$ and $m \geq 0$ the quantity

$$\llbracket K^\varepsilon \rrbracket_{\zeta;m}^{(\varepsilon)} \stackrel{\text{def}}{=} \max_{|k|_\mathfrak{s} \leq m} \sup_{z \notin P_0} \frac{\big|D_\varepsilon^k K^\varepsilon(z)\big|}{\|z\|_{\mathfrak{s},\varepsilon}^{(\zeta - |k|_\mathfrak{s}) \wedge 0}} , \tag{5.13}$$

where $z \in \mathbb{R} \times \Lambda_\varepsilon^d$, $k \in \mathbb{N}^{d+1}$ and $\|z\|_{\mathfrak{s},\varepsilon} \stackrel{\text{def}}{=} \|z\|_\mathfrak{s} \vee \varepsilon$.

By analogy with Remark 4.2.7, we always consider a sequence of functions parametrised by $\varepsilon = 2^{-N}$ with $N \in \mathbb{N}$, and we assume the bounds to hold for all ε with proportionality constants independent of ε. Thus, if $[\![K^\varepsilon]\!]^{(\varepsilon)}_{\zeta;m} < \infty$, then we will say that K^ε is of order ζ.

Remark 5.3.5. We stress the fact that by our assumptions the functions K^ε are defined also at the origin. In particular, K^ε can have a discontinuity at $t = 0$ and its time derivative behaves in the worst case as the Dirac delta function at the origin.

The following result provides bounds on products and convolutions $\star_\varepsilon$ defined in Section 4.1.1, of such functions.

Lemma 5.3.6. *Let functions K_1^ε and K_2^ε be of orders ζ_1 and ζ_2 respectively. Then we have the following results:*

- *If $\zeta_1, \zeta_2 \leq 0$, then $K_1^\varepsilon K_2^\varepsilon$ is of order $\zeta_1 + \zeta_2$ and for every $m \geq 0$ one has*

$$[\![K_1^\varepsilon K_2^\varepsilon]\!]^{(\varepsilon)}_{\zeta_1+\zeta_2;m} \lesssim [\![K_1^\varepsilon]\!]^{(\varepsilon)}_{\zeta_1;m} [\![K_2^\varepsilon]\!]^{(\varepsilon)}_{\zeta_2;m} . \tag{5.14}$$

  Moreover, of both K_1^ε and K_2^ε are continuous in the time variable on whole $\mathbb{R}$, then $K_1^\varepsilon K_2^\varepsilon$ is continuous as well.

- *If $\zeta_1 \wedge \zeta_2 > -|\mathfrak{s}|$ and $\bar\zeta \stackrel{\text{def}}{=} \zeta_1 + \zeta_2 + |\mathfrak{s}| \notin \mathbb{N}$, then $K_1^\varepsilon \star_\varepsilon K_2^\varepsilon$ is continuous in the time variable and one has the bound*

$$[\![K_1^\varepsilon \star_\varepsilon K_2^\varepsilon]\!]^{(\varepsilon)}_{\bar\zeta;m} \lesssim [\![K_1^\varepsilon]\!]^{(\varepsilon)}_{\zeta_1;m} [\![K_2^\varepsilon]\!]^{(\varepsilon)}_{\zeta_2;m} . \tag{5.15}$$

In all these estimates the proportionality constants depend only on the support of the functions K_i^ε and are independent of ε.

Proof of Lemma 5.3.6. The bound (5.14) follows from the Leibniz rule for the discrete derivative:

$$D_\varepsilon^k \big(K_1^\varepsilon K_2^\varepsilon\big)(z) = \sum_{l \leq k} \binom{k}{l} D_\varepsilon^l K_1^\varepsilon(z)\, D_\varepsilon^{k-l} K_2^\varepsilon\big(z + (0, \varepsilon l)\big) , \tag{5.16}$$

where $k, l \in \mathbb{N}^d$, as well as from the standard Leibniz rule in the time variable. The bound (5.15) can be proved similarly to [Hai14, Lem. 10.14], but using the

Leibniz rule (5.16), summation by parts for the discrete derivative and the fact that the products

$$(x)_{k,\varepsilon} \stackrel{\text{def}}{=} \prod_{i=1}^{d} \prod_{0\leq j<k_i} (x_i - \varepsilon j)$$

with $k \in \mathbb{N}^d$ play the role of polynomials for the discrete derivative.

When bounding the time derivative of $K_1^\varepsilon \star_\varepsilon K_2^\varepsilon$, we convolve in the worst case a function which behaves as Dirac's delta at the origin with another one which has a jump there (see Remark 5.3.5). This operation gives us a function whose derivative can have a jump at the origin, but is not Dirac's delta. This fact explains why $K_1^\varepsilon \star_\varepsilon K_2^\varepsilon$ is continuous in time. $\square$

The following lemma, whose proof is almost identical to that of [Hai14, Lem. 10.18], provides a bound on an increment of a singular function.

Lemma 5.3.7. *Let a function K^ε be of order $\zeta \leq 0$. Then for every $\kappa \in [0,1]$, $t \in \mathbb{R}$ and $x_1, x_2 \in \Lambda_\varepsilon^d$ one has*

$$|K^\varepsilon(t,x_1) - K^\varepsilon(t,x_2)| \lesssim |x_1 - x_2|^\kappa \Big(\|t,x_1\|_{\mathfrak{s},\varepsilon}^{\zeta-\kappa} + \|t,x_2\|_{\mathfrak{s},\varepsilon}^{\zeta-\kappa} \Big) [\![K^\varepsilon]\!]_{\zeta;1}^{(\varepsilon)} .$$

For a discrete singular function K^ε, we define the function $\mathscr{R}_\varepsilon K^\varepsilon$ by

$$(\mathscr{R}_\varepsilon K^\varepsilon)(\varphi) \stackrel{\text{def}}{=} \int_{\mathbb{R}\times\Lambda_\varepsilon^d} K^\varepsilon(z)(\varphi(z) - \varphi(0))\, dz\,, \tag{5.17}$$

for every compactly supported test function φ on $\mathbb{R}^{d+1}$. The following result can be proved similarly to [Hai14, Lem. 10.16] and using the argument from the proof of Lemma 5.3.6.

Lemma 5.3.8. *Let functions K_1^ε and K_2^ε be of orders ζ_1 and ζ_2 respectively with $\zeta_1 \in \big(-|\mathfrak{s}| - 1, -|\mathfrak{s}|\big]$ and $\zeta_2 \in \big(-2|\mathfrak{s}| - \zeta_1, 0\big]$. Then the function $\big(\mathscr{R}_\varepsilon K_1^\varepsilon\big) \star_\varepsilon K_2^\varepsilon$ is continuous in time of order $\bar{\zeta} \stackrel{\text{def}}{=} \zeta_1 + \zeta_2 + |\mathfrak{s}|$ and, for any $m \geq 0$, one has*

$$[\![\big(\mathscr{R}_\varepsilon K_1^\varepsilon\big) \star_\varepsilon K_2^\varepsilon]\!]_{\bar{\zeta};m}^{(\varepsilon)} \lesssim [\![K_1^\varepsilon]\!]_{\zeta_1;m}^{(\varepsilon)} [\![K_2^\varepsilon]\!]_{\zeta_2;m+\mathfrak{s}_0}^{(\varepsilon)} .$$

The following result shows how certain convolutions change singular functions. Its proof is similar to [Hai14, Lem. 10.17].

Lemma 5.3.9. *Let for some $\bar{\varepsilon} \in [\varepsilon, 1]$ the function $\psi^{\bar{\varepsilon},\varepsilon} : \mathbb{R} \times \Lambda_\varepsilon^d \to \mathbb{R}$ be smooth in the time variable, supported in the ball $B(0, R\bar{\varepsilon}) \subset \mathbb{R}^{d+1}$ for some $R \geq 1$, and satisfies*

$$\int_{\mathbb{R}\times\Lambda_\varepsilon^d} \psi^{\bar{\varepsilon},\varepsilon}(z)\, dz = 1\,, \qquad |D_\varepsilon^k \psi^{\bar{\varepsilon},\varepsilon}(z)| \lesssim \bar{\varepsilon}^{-|\mathfrak{s}|-|k|_\mathfrak{s}}\,,$$

for all $z \in \mathbb{R} \times \Lambda_\varepsilon^d$ and $k \in \mathbb{N}^{d+1}$, where the proportionality constant in the bound can depend on k. If K^ε is of order $\zeta \in (-|\mathfrak{s}|, 0)$, then for all $\kappa \in (0, 1]$ one has

$$\llbracket K^\varepsilon - K^\varepsilon \star_\varepsilon \psi^{\bar{\varepsilon},\varepsilon} \rrbracket_{\zeta-\kappa;m}^{(\varepsilon)} \lesssim \bar{\varepsilon}^\kappa \llbracket K^\varepsilon \rrbracket_{\zeta;m+\mathfrak{s}_0}^{(\varepsilon)}\,.$$

5.4 Proof of the convergence result

Using the results from the previous section, we are ready to prove Theorem 5.1.1. We start with an auxiliary result, providing a bound on the renormalised discrete model $\hat{Z}^\varepsilon$ defined at the end of Section 5.2. In what follows we will also consider its "approximations" defined in the following way: we take a function $\psi : \mathbb{R}^4 \to \mathbb{R}$ which is smooth, compactly supported and integrates to 1, and for some $\bar{\varepsilon} \in [\varepsilon, 1]$ we define

$$\psi^{\bar{\varepsilon}}(t, x) \stackrel{\text{def}}{=} \bar{\varepsilon}^{-5}\psi\big(\bar{\varepsilon}^{-2}t, \bar{\varepsilon}^{-1}x\big)\,. \tag{5.18}$$

Furthermore, we define the function

$$\psi^{\bar{\varepsilon},\varepsilon}(t, x) \stackrel{\text{def}}{=} \varepsilon^{-d} \int_{\mathbb{R}^d} \psi^{\bar{\varepsilon}}(t, y)\, \mathbf{1}_{|y-x|_\infty \leq \varepsilon/2}\, dy\,, \qquad (t, x) \in \mathbb{R} \times \Lambda_\varepsilon^d\,, \tag{5.19}$$

and the discrete noise $\xi^{\bar{\varepsilon},\varepsilon} \stackrel{\text{def}}{=} \psi^{\bar{\varepsilon},\varepsilon} \star_\varepsilon \xi^\varepsilon$, where ξ^ε is given in (5.4). Then we define the discrete model $\hat{Z}^{\bar{\varepsilon},\varepsilon}$ by substituting each occurrence of ξ^ε, $C_1^{(\varepsilon)}$ and $C_2^{(\varepsilon)}$ in the definition of $\hat{Z}^\varepsilon$ by $\xi^{\bar{\varepsilon},\varepsilon}$, $C_1^{(\bar{\varepsilon},\varepsilon)}$ and $C_2^{(\bar{\varepsilon},\varepsilon)}$ respectively, where the latter renormalisation constants are given by

$$C_1^{(\bar{\varepsilon},\varepsilon)} \stackrel{\text{def}}{=} \int_{\mathbb{R}\times\Lambda_\varepsilon^3} \big(K^{\bar{\varepsilon},\varepsilon}(z)\big)^2\, dz\,, \quad C_2^{(\bar{\varepsilon},\varepsilon)} \stackrel{\text{def}}{=} 2\int_{\mathbb{R}\times\Lambda_\varepsilon^3} \big(K^{\bar{\varepsilon},\varepsilon} \star_\varepsilon K^{\bar{\varepsilon},\varepsilon}\big)(z)^2\, K^\varepsilon(z)\, dz\,, \tag{5.20}$$

where $K^{\bar{\varepsilon},\varepsilon} \stackrel{\text{def}}{=} K^\varepsilon \star_\varepsilon \psi^{\bar{\varepsilon},\varepsilon}$. Then we have the following result:

Lemma 5.4.1. *In the described situation we have, for any $T > 0$, $p \geq 1$ and for* *sufficiently small values of $\delta > 0$ and $\theta > 0$, the following bounds*

$$\mathbb{E}\Big[\|\hat{Z}^\varepsilon\|^{(\varepsilon)}_{\delta,\gamma;T}\Big]^p \lesssim 1 \;, \qquad \mathbb{E}\Big[\|\hat{Z}^{\bar{\varepsilon},\varepsilon}; \hat{Z}^\varepsilon\|^{(\varepsilon)}_{\delta,\gamma;T}\Big]^p \lesssim \bar{\varepsilon}^{\theta p} \;, \tag{5.21}$$

uniformly in $\varepsilon \in (0, \bar{\varepsilon}]$ in the sense of Remark 4.2.7.

Proof. In order to prove this lemma, we will use Theorem 4.4.1. To this end, we will check that the discrete models satisfy the bounds (4.64) and the assumptions of Proposition 4.4.2 for the symbols $\tau \in \hat{\mathcal{F}}^-$ introduced in (5.8). Throughout the proof we will use the bounds

$$[\![K^\varepsilon]\!]^{(\varepsilon)}_{-3;m} < \infty \;, \qquad [\![K^{\bar{\varepsilon},\varepsilon} - K^\varepsilon]\!]^{(\varepsilon)}_{-3-\theta;m} \lesssim \bar{\varepsilon}^\theta \;, \tag{5.22}$$

for any $\theta \in (0, 1]$ and some $m > 0$ sufficiently large, which follow from Lemmas 4.3.4 and 5.3.9 and Remark 4.3.5 (one can see that the function $\psi^{\bar{\varepsilon},\varepsilon}$ defined in (5.19) satisfies all the assumptions of Lemma 5.3.9). Here, we have used the quantity defined in (5.13). Moreover, choosing $r > 0$ in Lemma 4.3.4 sufficiently large, we can make sure that the value of m can be taken large enough and all the bounds below make sense.

We start from the symbol Ψ. Since the renormalisation map M from (5.7) leaves Ψ invariant, the definition (5.9) yields $\hat{\Pi}^\varepsilon \Psi = \Pi^\varepsilon \Psi$, $\hat{\Gamma}^\varepsilon \Psi = \Gamma^\varepsilon \Psi$, $\hat{\Sigma}^\varepsilon \Psi = \Sigma^\varepsilon \Psi$, from which the bounds (4.64) follow trivially. Moreover, $\hat{\Pi}^\varepsilon \Psi$ can be represented as in (4.69) by the 1-st order Wiener integral with the respective kernel $(\mathcal{W}^{(\varepsilon;1)}\Psi)(z; z_1) = K^\varepsilon(z - z_1)$. The function (4.70) in this case is given by

$$\big(\mathcal{K}^{(\varepsilon;1)}\Psi\big)(z_1, z_2) = \big(K^\varepsilon \star_\varepsilon K^\varepsilon\big)(z_1 - z_2) \;. \tag{5.23}$$

Hence, we can use the bound (5.22) and Lemma 5.3.6 to get $[\![\mathcal{K}^{(\varepsilon;1)}\Psi]\!]^{(\varepsilon)}_{-1;\bar{m}} < \infty$ for some $\bar{m} > 0$. Moreover, $\mathcal{K}^{(\varepsilon;1)}\Psi$ is continuous in the time variable. One can see that these facts and the Taylor formula imply that the assumptions of Proposition 4.4.2 are satisfied with $\alpha = -\frac{1}{2}$ and any $\delta \in (0, 1)$, which yields the bounds (4.65) on $\hat{\Pi}^\varepsilon \Psi$ as soon as $|\Psi| < -\frac{1}{2}$.

Now, we will bound the difference of the two models $\hat{Z}^\varepsilon$ and $\hat{Z}^{\bar{\varepsilon},\varepsilon}$ applied to Ψ. The required bounds (4.64) on $\big(\hat{\Gamma}^\varepsilon - \hat{\Gamma}^{\bar{\varepsilon},\varepsilon}\big)\Psi$ and $\big(\hat{\Sigma}^\varepsilon - \hat{\Sigma}^{\bar{\varepsilon},\varepsilon}\big)\Psi$ follow trivially as before. Furthermore, the Wiener chaos expansion of $\hat{\Pi}^{\bar{\varepsilon},\varepsilon}\Psi$ is given by the 1-st order integral with the kernel $(\bar{\mathcal{W}}^{(\varepsilon;1)}\Psi)(z; z_1) = K^{\bar{\varepsilon},\varepsilon}(z - z_1)$, and the respective function (4.70) is given by

$$\big(\bar{\mathcal{K}}^{(\varepsilon;1)}\Psi\big)(z_1, z_2) = \big(K^{\bar\varepsilon,\varepsilon} \star_\varepsilon K^{\bar\varepsilon,\varepsilon}\big)(z_1 - z_2) \;.$$

Hence, using (5.23) and denoting $\delta\mathcal{K}^{(\varepsilon;1)} \stackrel{\text{def}}{=} \bar{\mathcal{K}}^{(\varepsilon;1)} - \mathcal{K}^{(\varepsilon;1)}$, we can write

$$\delta\mathcal{K}^{(\varepsilon;1)}\Psi = (K^{\bar\varepsilon,\varepsilon} - K^\varepsilon) \star_\varepsilon K^{\bar\varepsilon,\varepsilon} + K^\varepsilon \star_\varepsilon (K^{\bar\varepsilon,\varepsilon} - K^\varepsilon) \;. \tag{5.24}$$

Exploiting (5.22) and Lemma 5.3.6, we bound each term in (5.24) separately and obtain

$$\llbracket \delta\mathcal{K}^{(\varepsilon;1)}\Psi \rrbracket^{(\varepsilon)}_{-1-2\theta;\bar m} \lesssim \varepsilon^{2\theta} \;,$$

and the function $\delta\mathcal{K}^{(\varepsilon;1)}\Psi$ is continuous in the time variable. This means that as soon as $\theta > 0$ is small enough, the assumptions of the second part of Proposition 4.4.2 are satisfied with some $\delta > 0$ sufficiently small, which gives bounds of the type (4.65) on $\big(\hat\Pi^{\bar\varepsilon,\varepsilon} - \hat\Pi^\varepsilon\big)\Psi$ with the proportionality constants of order $\varepsilon^{2\theta}$.

It follows from the Wick lemma [Nua06, Prop. 1.1.2], the definitions of the renormalisation map M^ε and the renormalisation constant $C_1^{(\varepsilon)}$ in (5.11) that one has the identity $\hat\Pi^\varepsilon\Psi^2 = (\Pi^\varepsilon\Psi)^{\diamond 2}$, where $\diamond$ is the Wick power [Nua06]. Hence, the expansion (4.69) of $\hat\Pi^\varepsilon\Psi^2$ is given only by the second order integral with the kernel

$$(\mathcal{W}^{(\varepsilon;2)}\Psi^2)(z; z_1, z_2) = K^\varepsilon(z - z_1)K^\varepsilon(z - z_2) \;.$$

The respective function (4.70) for this kernel can be written as

$$\big(\mathcal{K}^{(\varepsilon;2)}\Psi^2\big)(z_1, z_2) = \big(\mathcal{K}^{(\varepsilon;1)}\Psi\big)^2(z_1, z_2) \;, \tag{5.25}$$

which satisfies $\llbracket \mathcal{K}^{(\varepsilon;2)}\Psi^2 \rrbracket^{(\varepsilon)}_{-2;\bar m} < \infty$ and $\mathcal{K}^{(\varepsilon;2)}\Psi^2$ is continuous in time (here, we have used the bound on $\mathcal{K}^{(\varepsilon;1)}\Psi$ obtained above and Lemma 5.3.6). As before, these properties guarantee that the assumptions of Proposition 4.4.2 are satisfied for Ψ^2 and $\delta > 0$ sufficiently small, and the bounds (4.65) hold. Defining the respective function $\bar{\mathcal{K}}^{(\varepsilon;2)}\Psi^2$ for $\hat\Pi^{\bar\varepsilon,\varepsilon}\Psi^2$ by replacing $\mathcal{K}^{(\varepsilon;1)}\Psi$ by $\bar{\mathcal{K}}^{(\varepsilon;1)}\Psi$ on the right-hand side of (5.25), we obtain

$$\delta\mathcal{K}^{(\varepsilon;2)}\Psi^2 = (\delta\mathcal{K}^{(\varepsilon;1)}\Psi)(\bar{\mathcal{K}}^{(\varepsilon;1)}\Psi + \mathcal{K}^{(\varepsilon;1)}\Psi) \;.$$

Table 5.1: Components of diagrams

Component	Description
$z\bullet$	a variable z in $\mathbb{R} \times \Lambda_\varepsilon^3$
$\bullet$ (grey)	a variable in $\mathbb{R} \times \Lambda_\varepsilon^3$ which is integrated out
$z_1 \bullet\!\!\longrightarrow\!\!\bullet\, z_2$	the function $K^\varepsilon(z_2 - z_1)$
$z_1 \bullet\!\cdots\!\!\rightarrow\!\!\bullet\, z_2$	the function $K^{\bar{\varepsilon},\varepsilon}(z_2 - z_1)$
$z_1 \bullet\!\!\rightsquigarrow\!\!\bullet\, z_2$	the function $(K^{\bar{\varepsilon},\varepsilon} - K^\varepsilon)(z_2 - z_1)$

Combining the bounds on the functions for Ψ obtained above with Lemma 5.3.6, we get

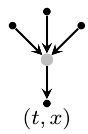
$$\vert\!\vert\!\vert \delta \mathcal{K}^{(\varepsilon;2)} \Psi^2 \vert\!\vert\!\vert^{(\varepsilon)}_{-2-2\theta;\bar{m}} \lesssim \bar{\varepsilon}^{2\theta} ,$$

for sufficiently small $\theta > 0$. As before, we apply Proposition 4.4.2 to conclude that there are $\delta > 0$ and $\theta > 0$ such that the bounds (4.65) hold for $\big(\hat{\Pi}^{\bar{\varepsilon},\varepsilon} - \hat{\Pi}^{\varepsilon}\big)\Psi^2$ with the proportionality constants of order $\bar{\varepsilon}^{2\theta}$.

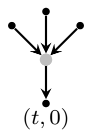
The required bounds for the symbol $\Psi^2 \mathbb{X}_i$ follow immediately from the results of the previous paragraph and the standard action of discrete models on polynomials.

In order to proceed, we will follow the idea of [Hai14, Sec. 10.5] and represent kernels by diagrams, whose components are described in Table 5.1. With these diagrams at hand, we turn to the symbol $\bar{\Psi}$. Using the renormalisation constant $C_1^{(\varepsilon)}$, defined in (5.11), and the Wick's lemma, we conclude from (5.10) that $\hat{\Pi}^\varepsilon \bar{\Psi}$ is in the third homogeneous Wiener chaos, and can be written as (4.69) with the respective kernel

$$\big(\mathcal{W}^{(\varepsilon;3)} \bar{\Psi}\big)(t,x) = \quad (5.26)$$

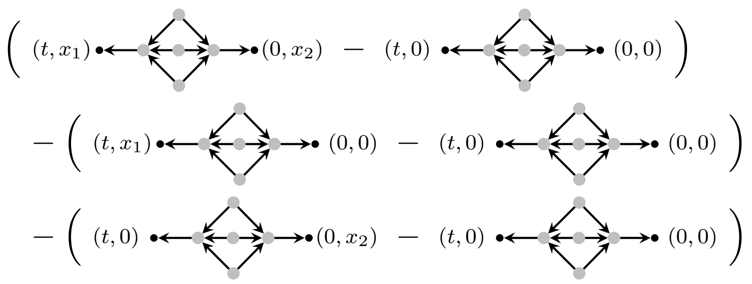


$-$



In fact, for each fixed z, the kernel $\big(\mathcal{W}^{(\varepsilon;3)} \bar{\Psi}\big)(z)$ is a function of three variables, whose values should be assigned to the leaves of each tree in the diagram (5.26). In order to simplify the notation, we will not write these variables in the sequel. Moreover, the product in $H_\varepsilon^{\otimes 3}$ in the definition of the function $\mathcal{K}^{(\varepsilon;3)} \bar{\Psi}$ in (4.70) is equivalent to pairing and integrating out the respective variables for two copies of the diagram (5.26). Precisely, the function $\mathcal{K}^{(\varepsilon;3)} \bar{\Psi}$ for this kernel can be represented in the following way

$$\big(\mathcal{K}^{(\varepsilon;3)}\bar{\Psi}\big)_t(x_1,x_2) =$$



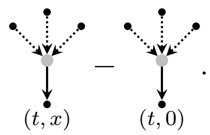
Bounding each of these brackets separately by applying recursively Lemma 5.3.6 and the Taylor formula in the spatial variable, we obtain

$$\big|D_t^\ell\big(\mathcal{K}^{(\varepsilon;3)}\bar{\Psi}\big)_t(x_1,x_2)\big| \lesssim \|t,x_1-x_2\|_{\mathfrak{s},\varepsilon}^{\zeta-2\ell} + \|t,x_1\|_{\mathfrak{s},\varepsilon}^{\zeta-2\ell} + \|t,x_2\|_{\mathfrak{s},\varepsilon}^{\zeta-2\ell}\;,$$

for $\ell \in \{0,1\}$ and any $\zeta < 1$, where the estimate with $\ell = 1$ fails to hold only for $t = 0$. Moreover, $\mathcal{K}^{(\varepsilon;3)}\bar{\Psi}$ is continuous in the time variable. The assumptions of Proposition 4.4.2 with $\alpha < \frac{1}{2}$ and $\delta > 0$ sufficiently small follow from these bounds which implies that (4.65) hold on $\hat{\Pi}^\varepsilon\bar{\Psi}$.

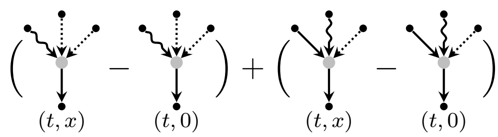
One can see that the map $\hat{\Pi}^{\bar{\varepsilon},\varepsilon}\bar{\Psi}$ can be represented by (4.69) with the kernel $\bar{\mathcal{W}}^{(\varepsilon;3)}\bar{\Psi}$ which is described by the following diagram

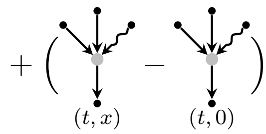
$$\big(\bar{\mathcal{W}}^{(\varepsilon;3)}\bar{\Psi}\big)(t,x) =$$



Then using the notation from Table 5.1 we can write

$$\big(\bar{\mathcal{W}}^{(\varepsilon;3)}\bar{\Psi} - \mathcal{W}^{(\varepsilon;3)}\bar{\Psi}\big)(t,x) =$$





$$\stackrel{\text{def}}{=} \sum_{i=1}^{3}\big(\mathcal{W}_i^{(\varepsilon;3)}\bar{\Psi}\big)(t,x)\;.$$

Defining now for these new kernels the functions

$$\big(\mathcal{K}_{i,j}^{(\varepsilon;3)}\bar{\Psi}\big)_t(x_1,x_2) \stackrel{\text{def}}{=} \langle\big(\mathcal{W}_i^{(\varepsilon;3)}\bar{\Psi}\big)(t,x_1), \big(\mathcal{W}_j^{(\varepsilon;3)}\bar{\Psi}\big)(0,x_2)\rangle_{H_\varepsilon^{\otimes 3}}\;,$$

and treating them in the same way as the function $\mathcal{K}^{(\varepsilon;3)}\bar{\Psi}$ above, we obtain

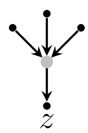
$$\left|D_t^\ell\big(\mathcal{K}_{i,j}^{(\varepsilon;3)}\bar{\Psi}\big)_t(x_1,x_2)\right| \lesssim \varepsilon^{2\theta}\big(\|t,x_1-x_2\|_{\mathfrak{s},\varepsilon}^{\zeta-2\theta-2\ell} + \|t,x_1\|_{\mathfrak{s},\varepsilon}^{\zeta-2\theta-2\ell} \\ + \|t,x_2\|_{\mathfrak{s},\varepsilon}^{\zeta-2\theta-2\ell}\big)\;,$$

for $\ell \in \{0,1\}$, any $\zeta < 1$ and every $\theta > 0$ sufficiently small, where as before the estimate with $\ell = 1$ fails to hold only for $t = 0$. The multiplier $\varepsilon^{2\theta}$ appears in this bound, because exactly two kernels $K^{\bar{\varepsilon},\varepsilon} - K^\varepsilon$ are involved in the definition of each $\mathcal{K}_{i,j}^{(\varepsilon;3)}\bar{\Psi}$. Exploiting as before Proposition 4.4.2 with $\theta > 0$ and $\delta > 0$ sufficiently small, we obtain the required bounds (4.65) on $(\hat{\Pi}^{\bar{\varepsilon},\varepsilon} - \hat{\Pi}^\varepsilon)\bar{\Psi}$.

Now, we will prove that the bounds (4.64) hold for the symbol $\bar{\Psi}$. It is easy to see that the function $\bar{\psi}^\varepsilon$ involved in the definition (5.10) is given by

$$\bar{\psi}^\varepsilon(z) = I_3^\varepsilon\Big(\big(Q^{(\varepsilon;3)}\bar{\Psi}\big)(z)\Big)\;,$$

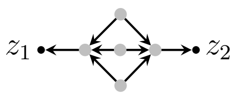
where I_3^ε is the Wiener integral as in (4.69), and the kernel $Q^{(\varepsilon;3)}\bar{\Psi}$ can be written as

$\big(Q^{(\varepsilon;3)}\bar{\Psi}\big)(z) =$  .

Defining for this kernel the respective function

$$\big(\mathcal{L}^{(\varepsilon;3)}\bar{\Psi}\big)(z_1,z_2) \stackrel{\text{def}}{=} \langle\big(Q^{(\varepsilon;3)}\bar{\Psi}\big)(z_1), \big(Q^{(\varepsilon;3)}\bar{\Psi}\big)(z_2)\rangle_{H_\varepsilon^{\otimes 3}}\;,$$

we can write it using diagrams in the following way

$\big(\mathcal{L}^{(\varepsilon;3)}\bar{\Psi}\big)(z_1,z_2) = \big(\mathcal{L}^{(\varepsilon;3)}\bar{\Psi}\big)(z_1-z_2) =$  .

Thus, applying Lemma 5.3.6 and the Taylor's formula, we obtain the bound

$$\left|\big(\mathcal{L}^{(\varepsilon;3)}\bar{\Psi}\big)(z) - \big(\mathcal{L}^{(\varepsilon;3)}\bar{\Psi}\big)(0)\right| \lesssim \|z\|_{\mathfrak{s}}^\zeta\;,$$

for any $\zeta < 1$. Since equivalence of moments for Wiener integrals holds and we have the following bound on the second moment

$$\mathbb{E}\big[\bar{\psi}^\varepsilon(z_1) - \bar{\psi}^\varepsilon(z_2)\big]^2 \lesssim \big|\big(\mathcal{L}^{(\varepsilon;3)}\bar{\Psi}\big)(z_1 - z_2) - \big(\mathcal{L}^{(\varepsilon;3)}\bar{\Psi}\big)(0)\big| \lesssim \|z_1 - z_2\|_{\mathfrak{s}}^{\zeta} ,$$

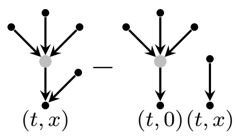
we can use the Kolmogorov continuity criterion to conclude that

$$\mathbb{E}\left[\sup_{z_1 \neq z_2} \frac{\big|\bar{\psi}^\varepsilon(z_1) - \bar{\psi}^\varepsilon(z_2)\big|}{\|z_1 - z_2\|_{\mathfrak{s}}^{\zeta}}\right] \lesssim 1 ,$$

holds for every $\zeta < \frac{1}{2}$. From this bound and definition (5.10), the estimate (4.64) follows immediately. The respective bound on the difference of the two discrete models can be proved as before.

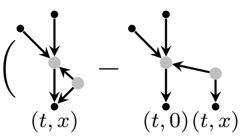
Now, we will consider the symbol $\Psi\bar{\Psi}$. One can use the Wick lemma to see that $\hat{\Pi}^\varepsilon \Psi\bar{\Psi}$ is in the fourth inhomogeneous Wiener chaos and we will describe each integral in the expansion (4.69) separately. Its contribution to the fourth Wiener chaos is given by

$$\big(\mathcal{W}^{(\varepsilon;4)}\Psi\bar{\Psi}\big)(t,x) = \qquad (5.27)$$



The term from the second Wiener chaos is described by the kernel

$$\big(\mathcal{W}^{(\varepsilon;2)}\Psi\bar{\Psi}\big)(t,x) = 3\Big(\quad - \quad \Big) \stackrel{\text{def}}{=} 3\Big(\mathcal{W}_1^{(\varepsilon;2)}\Psi\bar{\Psi} + \mathcal{W}_2^{(\varepsilon;2)}\Psi\bar{\Psi}\Big)(t,x) . \qquad (5.28)$$



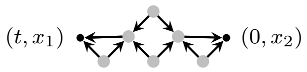
Here, the diagrams of $\mathcal{W}^{(\varepsilon;2)}\Psi\bar{\Psi}$ are obtained by pairing and integrating out any two leaves of the diagrams (5.27), which is the result of the Wick lemma. We don't consider pairings of the other leaves in (5.27), because these terms are removed by renormalisation (5.7).

We first derive the required bounds for $\mathcal{W}^{(\varepsilon;4)}\Psi\bar{\Psi}$, whose respective function (4.70) can be written as

$$\mathcal{K}^{(\varepsilon;4)}\Psi\bar{\Psi} = \big(\mathcal{K}^{(\varepsilon;1)}\Psi\big)\,\big(\mathcal{K}^{(\varepsilon;3)}\bar{\Psi}\big) , \qquad (5.29)$$

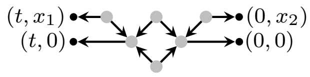
where the functions $\mathcal{K}^{(\varepsilon;1)}\Psi$ and $\mathcal{K}^{(\varepsilon;3)}\bar{\Psi}$ have been defined above. Using the bounds on these functions obtained above and the Leibniz rule in the time variable, we get the bounds on $\mathcal{K}^{(\varepsilon;4)}\Psi\bar{\Psi}$ of the type (4.72) for any $\zeta < 0$ and a sufficiently small value of $\delta > 0$.

In the case of the kernel $\mathcal{W}_1^{(\varepsilon;2)}\Psi\bar{\Psi}$, the function $\mathcal{K}_1^{(\varepsilon;2)}\Psi\bar{\Psi}$ is given by

$$\big(\mathcal{K}_1^{(\varepsilon;2)}\Psi\bar{\Psi}\big)_t(x_1, x_2) =$$  .

Then it follows from Lemma 5.3.6 that $\|\mathcal{K}_1^{(\varepsilon;2)}\Psi\bar{\Psi}\|_{\zeta;m}^{(\varepsilon)} \lesssim 1$, for every $\zeta < 0$, and this function is continuous in time. As before, these facts imply the bounds of the type (4.72) with $\delta > 0$ sufficiently small.

The function (4.70), determined by the kernel $\mathcal{W}_2^{(\varepsilon;2)}\Psi\bar{\Psi}$ from (5.27), can be written as

$$\big(\mathcal{K}_2^{(\varepsilon;2)}\Psi\bar{\Psi}\big)_t(x_1, x_2) =$$  .

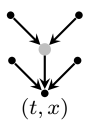
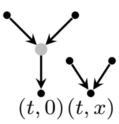
Applying consecutively Lemmas 5.3.7 and 5.3.6 we conclude that for every $\zeta < 0$ and $\ell \in \{0, 1\}$ one has the bound

$$|D_t^\ell(\mathcal{K}_2^{(\varepsilon;2)}\Psi\bar{\Psi})_t(x_1, x_2)| \lesssim |x_1 - x_2|^{\zeta - 2\ell}\;,$$

and the function $\mathcal{K}_2^{(\varepsilon;2)}\Psi\bar{\Psi}$ is continuous in time. This as usual implies the bounds of the type (4.72) for $\delta > 0$ small enough.

Combining all the bounds from above for the symbol $\Psi\bar{\Psi}$ and using Proposition 4.4.2, we conclude that the estimates (4.65) hold for $\Psi\bar{\Psi}$. The required bounds on the difference of the two discrete models can be proved in the same way as before.

Now, we will consider the symbol $\tau = \mathcal{I}(\Psi^2)\Psi^2$. The Wick's lemma yields that $\hat{\Pi}^\varepsilon\tau$ belongs to the fourth inhomogeneous Wiener chaos, and its contribution to the 4-th Wiener chaos is described by the kernel

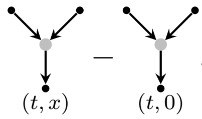
$$\big(\mathcal{W}^{(\varepsilon;4)}\tau\big)(t, x) =$$  $-$  . (5.30)

Denoting for brevity the symbol $\bar{\tau} = \mathcal{I}(\Psi^2)$, we can write the respective function (4.70) as

$$\mathcal{K}^{(\varepsilon;4)}\tau = \big(\mathcal{K}^{(\varepsilon;1)}\Psi\big)^2 \big(\mathcal{K}^{(\varepsilon;2)}\bar{\tau}\big)\;,$$

where $\mathcal{K}^{(\varepsilon;1)}\Psi$ is defined above and $\mathcal{K}^{(\varepsilon;2)}\bar{\tau}$ is given by (4.70) for the kernel

$$\big(\mathcal{W}^{(\varepsilon;2)}\bar{\tau}\big)(t,x) =$$



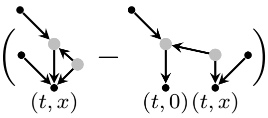
One can conclude from Lemma 5.3.6 and the Taylor formula that the function $\mathcal{K}^{(\varepsilon;2)}\bar{\tau}$ can be bounded by

$$|D_t^\ell\big(\mathcal{K}^{(\varepsilon;2)}\bar{\tau}\big)_t(x_1,x_2)| \lesssim \|t, x_1 - x_2\|_{\mathfrak{s},\varepsilon}^{\zeta-2\ell} + \|t, x_1\|_{\mathfrak{s},\varepsilon}^{\zeta-2\ell} + \|t, x_2\|_{\mathfrak{s},\varepsilon}^{\zeta-2\ell}\;,$$

for $\ell \in \{0,1\}$ and for every $\zeta < 2$, where as usual the time derivative can fail to exist only on $t = 0$. Furthermore, $\mathcal{K}^{(\varepsilon;2)}\bar{\tau}$ is continuous in the time variable. Combining these properties with the bound on $\mathcal{K}^{(\varepsilon;1)}\Psi$ obtained above, we get as usual an estimate on $\mathcal{K}^{(\varepsilon;4)}\tau$ of the type (4.72) for every $\alpha < 0$ and $\delta > 0$ small enough.

Using the renormalisation map (5.7) and the Wick's lemma, we conclude that the contribution of $\hat{\Pi}^\varepsilon\tau$ to the 2-nd Wiener chaos is given by the kernel

$$\big(\mathcal{W}^{(\varepsilon;2)}\tau\big)(t,x) = 4\Big($$

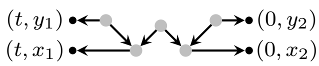


As before, we have got this diagrams by pairing two leaves in (5.30), where the other variants of pairings are excluded by renormalisation. Hence, the respective function (4.70) for this kernel can be written as

$$\big(\mathcal{K}^{(\varepsilon;2)}\tau\big)_t(x_1,x_2) = \big(\mathcal{K}^{(\varepsilon;1)}\Psi\big)_t(x_1,x_2)\,\big(\delta^{(2)}_{x_1,x_2} Q_{t,x_1,x_2}\big)\;,$$

where the function $\mathcal{K}^{(\varepsilon;1)}\Psi$ was defined before, the function Q_{t,x_1,x_2} is given by

$$Q_{t,x_1,x_2}(y_1,y_2) \stackrel{\text{def}}{=}$$



and the operator $\delta^{(2)}$ acts on a function f of two variables from $\mathbb{R}^3$ by

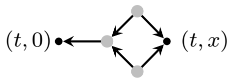
$$\delta^{(2)}_{x_1,x_2} f \stackrel{\text{def}}{=} f(x_1,x_2) - f(x_1,0) - f(0,x_2) + f(0,0)\;. \tag{5.31}$$

Applying consecutively Lemmas 5.3.7 and 5.3.6, we obtain the bound

$$|D_t^\ell \delta^{(2)}_{x_1,x_2} Q_{t,x_1,x_2}| \lesssim \|t, x_1 - x_2\|_{\mathfrak{s},\varepsilon}^{\zeta-2\ell} + \|t, x_1\|_{\mathfrak{s},\varepsilon}^{\zeta-2\ell} + \|t, x_2\|_{\mathfrak{s},\varepsilon}^{\zeta-2\ell} ,$$

for $\ell \in \{0, 1\}$ and for every $\zeta < 1$. Combining this result with the bound on $\mathcal{K}^{(\varepsilon;1)}\Psi$ obtained above, we get estimates of the type (4.72) on $\mathcal{K}^{(\varepsilon;2)}\tau$ with $\alpha < 0$ and $\delta > 0$ small enough.

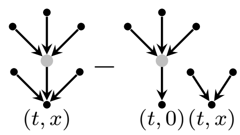
In order to describe the contribution of $\hat{\Pi}^\varepsilon\tau$ to the 0-th Wiener chaos, we use the renormalisation constant $C_2^{(\varepsilon)}$ defined in (5.11) and obtain the kernel

$$\big(\mathcal{W}^{(\varepsilon;0)}\tau\big)(t,x) = -2\Big($$  $$\Big) .$$

Applying Lemma 5.3.6, we get $|D_t^\ell(\mathcal{W}^{(\varepsilon;0)}\tau)(z)| \lesssim \|z\|_{\mathfrak{s},\varepsilon}^{\zeta-2\ell}$, for every $\zeta < 0$ and $\ell \in \{0, 1\}$. The bounds of the type (4.72) with $\alpha < 0$ and some $\delta > 0$ for the respective function (4.70) follows from this immediately.

Combining all the bounds obtained above for the symbol $\tau = \mathcal{I}(\Psi^2)\Psi^2$ and applying Proposition 4.4.2, we get the estimates (4.65). The bounds on the difference of the two discrete models can be obtained in the usual manner.

We now turn to the last symbol $\tau = \Psi^2\bar{\Psi}$. Wick's lemma yields that $\hat{\Pi}^\varepsilon\tau$ belongs to the fifth inhomogeneous Wiener chaos, whose fifth order Wiener integral is described by the kernel

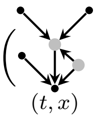
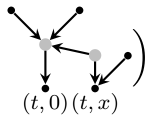
$$\big(\mathcal{W}^{(\varepsilon;5)}\tau\big)(t,x) =$$  $$. \qquad (5.32)$$

Then the respective function (4.70) for this kernel is given by

$$\big(\mathcal{K}^{(\varepsilon;5)}\tau\big)_t(x_1,x_2) = \big(\mathcal{K}^{(\varepsilon;1)}\Psi\big)_t^2(x_1,x_2)\,\big(\mathcal{K}^{(\varepsilon;3)}\bar{\Psi}\big)_t ,$$

where $\mathcal{K}^{(\varepsilon;1)}\Psi$ and $\mathcal{K}^{(\varepsilon;3)}\bar{\Psi}$ have been defined above. The bounds of the type (4.72) with $\alpha < -\frac{1}{2}$ now follow from the estimates on each multiplier obtained above.

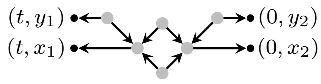
The contribution of $\hat{\Pi}^\varepsilon \tau$ to the 3-rd Wiener chaos is determined by the kernel

$$\big(\mathcal{W}^{(\varepsilon;3)}\tau\big)(t,x) = 6\Big($$  $$-$$  $$\Big)\;,$$

which is obtained by pairing two leaves in the diagram (5.32). The respective function (4.70) for this kernel can be written as

$$\big(\mathcal{K}^{(\varepsilon;3)}\tau\big)_t(x_1,x_2) = \big(\mathcal{K}^{(\varepsilon;1)}\Psi\big)_t(x_1,x_2)\,\big(\delta^{(2)}_{x_1,x_2}\tilde{Q}_{t,x_1,x_2}\big)\;,$$

where $\mathcal{K}^{(\varepsilon;1)}\Psi$ is defined above, the operator $\delta^{(2)}$ is from (5.31), and

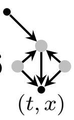
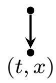
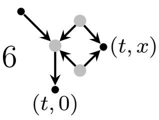
$$\tilde{Q}_{t,x_1,x_2}(y_1,y_2) =$$  $$.$$

Using Lemmas 5.3.6 and 5.3.7 we can get the following bound

$$|D_t^\ell \delta^{(2)}_{x_1,x_2}\tilde{Q}_{t,x_1,x_2}| \lesssim \|0,x_1-x_2\|_{\mathfrak{s},\varepsilon}^{-\kappa}\big(\|t,x_1-x_2\|_{\mathfrak{s},\varepsilon}^{\kappa-2\ell} + \|t,x_1\|_{\mathfrak{s},\varepsilon}^{\kappa-2\ell} + \|t,x_2\|_{\mathfrak{s},\varepsilon}^{\kappa-2\ell}\big)\;,$$

for every $\kappa > 0$ and $\ell \in \{0,1\}$. Combining these bounds with the bounds on $\mathcal{K}^{(\varepsilon;1)}\Psi$ proved above, we obtain the estimates of the type (4.72) on $\mathcal{K}^{(\varepsilon;3)}\tau$ with $\alpha < -\frac{1}{2}$ and sufficiently small $\delta > 0$.

It remains to bound the component of $\hat{\Pi}^\varepsilon\tau$ from the first Wiener chaos. The Wick's lemma implies that it is given by the kernel

$$\big(\mathcal{W}^{(\varepsilon;1)}\tau\big)(t,x) = \Big(6$$  $$-\,3C_2^{(\varepsilon)}$$  $$\Big) - 6$$ 

$$\stackrel{\text{def}}{=} 6\big(\mathcal{W}_1^{(\varepsilon;1)}\tau - \mathcal{W}_2^{(\varepsilon;1)}\tau\big)(t,x)\;.$$

It follows from the choice of $C_2^{(\varepsilon)}$ in (5.11) that we can write

$$\big(\mathcal{W}_1^{(\varepsilon;1)}\tau\big)(z;z_1) = \big((\mathscr{R}_\varepsilon L^\varepsilon)\star_\varepsilon K^\varepsilon\big)(z-z_1)\;,$$

where $L^\varepsilon(z) = \big(\mathcal{K}^{(\varepsilon;1)}\Psi\big)^2(z)K^\varepsilon(z)$, and $\mathscr{R}_\varepsilon$ is defined in (5.17). Then we conclude from Lemmas 5.3.8 and 5.3.6 that the respective function (4.70) defined for this kernel satisfies $[\![\mathcal{K}_1^{(\varepsilon;1)}\tau]\!]^{(\varepsilon)}_{\zeta;\bar{m}} \lesssim 1$ for every $\zeta < -1$. We can bound the function

$\mathcal{K}_2^{(\varepsilon;1)}\tau$ corresponding to the kernel $\mathcal{W}_2^{(\varepsilon;1)}\tau$ using Lemma 5.3.6 by

$$|D_t^\ell \big(\mathcal{K}_2^{(\varepsilon;1)}\tau\big)_t(x_1, x_2)| \lesssim \|0, x_1 - x_2\|_{\mathfrak{s},\varepsilon}^\zeta \big(\|t, x_1 - x_2\|_{\mathfrak{s},\varepsilon}^{\zeta-2\ell} + \|t, x_1\|_{\mathfrak{s},\varepsilon}^{\zeta-2\ell} + \|t, x_2\|_{\mathfrak{s},\varepsilon}^{\zeta-2\ell}\big) ,$$

for any $\zeta < -\frac{1}{2}$ and $\ell \in \{0, 1\}$, which gives us the bounds of the type (4.72) with $\alpha < -\frac{1}{2}$. Applying Proposition 4.4.2, we conclude that the bounds (4.65) hold for the symbol $\tau = \Psi^2\bar{\Psi}$. We can bound the difference of the models applied to $\Psi^2\bar{\Psi}$ in the usual manner, which finishes the proof. □

Let us define the mollified noise $\xi^{\bar{\varepsilon},0} \stackrel{\text{def}}{=} \xi \star \psi^{\bar{\varepsilon}}$ via the function $\psi^{\bar{\varepsilon}}$ introduced in (5.18), and let $\tilde{Z}^{(\bar{\varepsilon},0)}$ and $\tilde{Z}$ be the models on $\mathscr{T}$ built in [Hai14, Thm. 10.22] via the noises $\xi^{\bar{\varepsilon},0}$ and ξ respectively. We will be interested only in their restrictions to the truncated regularity structure $\hat{\mathscr{T}}$. It follows from the proof of the latter theorem that we are exactly in the setting of Section 4.4.1, and we can define respective inhomogeneous models $\hat{Z}^{\bar{\varepsilon},0}$ and $\hat{Z}$ on $\hat{\mathscr{T}}$ as in (4.78) and (4.79). Furthermore, we have the following bounds:

Lemma 5.4.2. *In the described context, the following bounds*

$$\mathbb{E}\Big[\|\hat{Z}\|_{\delta,\gamma;T}\Big]^p \lesssim 1\,, \qquad \mathbb{E}\Big[\|\hat{Z}^{\bar{\varepsilon},0}; \hat{Z}\|_{\delta,\gamma;T}\Big]^p \lesssim \bar{\varepsilon}^{\theta p}\,, \tag{5.33}$$

hold uniformly in $\bar{\varepsilon} \in (0, 1]$, for any $T > 0$, $p \geq 1$ and for sufficiently small values of $\delta > 0$ and $\theta > 0$.

Proof. It follows immediately from Lemma 3.3.3 that the singular part K of the heat kernel satisfies $[\![K]\!]_{-3;r} \leq C$, for arbitrary but fixed $r > 0$, where we have used the norm (5.12). Thus, this lemma can be proved using the exact expansions of the type (4.78) obtained for the elements of the models in the proof of [Hai14, Thm. 10.22] and repeating the same calculations as in the proof of Lemma 5.4.1, but using the continuous results from Section 5.3.1 instead of their discrete counterparts. □

With these results at hand we now provide a proof of Theorem 5.1.1.

Proof of Theorem 5.1.1. Using Theorem 3.3.11 and Lemmas 5.4.2 and 5.2.1, we define the solution Φ to the equation (3.2) as in Definition 3.3.12 by solving the respective abstract equation (3.50) with the inhomogeneous model $\hat{Z}$ from Lemma 5.4.2.

Furthermore, for every $K > 0$ we define the following stopping time:

$$\tau_K \stackrel{\text{def}}{=} \inf\bigl\{T > 0 : \|\Phi\|_{\mathcal{C}^{\delta,\alpha}_{\bar\eta,T}} \geq K\bigr\} ,$$

where the values of δ, α and $\bar\eta$ are as in the statement of the theorem. Then we have the limit in probability $\lim_{K\to\infty} \tau_K = T_\star$, where $T_\star$ is the random living time of Φ. Our aim is now to prove that

$$\lim_{K\to\infty} \lim_{\varepsilon\to 0} \mathbb{P}\Bigl[\|\Phi; \Phi^\varepsilon\|^{(\varepsilon)}_{\mathcal{C}^{\delta,\alpha}_{\bar\eta,\tau_K}} \geq c\Bigr] = 0 , \tag{5.34}$$

for every constant $c > 0$. Then the claim (5.5) will follow after choosing T_ε as a suitable diagonal sequence.

Throughout this proof we will use the objects defined in Section 5.2. In order to prove the claim, we proceed as in [Par77] and introduce intermediate equations driven by a smooth noise. Precisely, for a function $\psi^{\bar\varepsilon}$ as in (5.18) we define the mollified noise $\xi^{\bar\varepsilon,0} \stackrel{\text{def}}{=} \xi \star \psi^{\bar\varepsilon}$. Then we denote by $\Phi^{\bar\varepsilon,0}$ the global solution of the stochastic PDE driven by the smooth noise

$$\partial_t \Phi^{\bar\varepsilon,0} = \Delta \Phi^{\bar\varepsilon,0} - \bigl(\bigl(\Phi^{\bar\varepsilon,0}\bigr)^2 - C^{(\bar\varepsilon,0)}\bigr)\Phi^{\bar\varepsilon,0} + \xi^{\bar\varepsilon,0} , \qquad \Phi^{\bar\varepsilon,0}(0,\cdot) = \Phi_0(\cdot) ,$$

where $C^{(\bar\varepsilon,0)} \stackrel{\text{def}}{=} 3C_1^{(\bar\varepsilon,0)} - 9C_2^{(\bar\varepsilon,0)}$ is a renormalisation constant with

$$C_1^{(\bar\varepsilon,0)} \stackrel{\text{def}}{=} \int_{\mathbb{R}^4} \bigl(K_{\bar\varepsilon}(z)\bigr)^2\, dz , \qquad C_2^{(\bar\varepsilon,0)} \stackrel{\text{def}}{=} 2\int_{\mathbb{R}^4} \bigl(K_{\bar\varepsilon} \star K_{\bar\varepsilon}\bigr)(z)^2\, K(z)\, dz ,$$

and $K_{\bar\varepsilon} \stackrel{\text{def}}{=} K \star \psi^{\bar\varepsilon}$. As it was mentioned in Remark 3.3.13, the process $\Phi^{\bar\varepsilon,0}$ can be expressed as a solution to an abstract problem (3.50) with the inhomogeneous model $\hat Z^{\bar\varepsilon,0}$ from Lemma 5.4.2.

In order to discretise the noise $\xi^{\bar\varepsilon,0}$, we use the function $\psi^{\bar\varepsilon,\varepsilon}$ defined in (5.19), and set $\xi^{\bar\varepsilon,\varepsilon} \stackrel{\text{def}}{=} \psi^{\bar\varepsilon,\varepsilon} \star_\varepsilon \xi^\varepsilon$, where ξ^ε is given in (5.4). Let $\Phi^{\bar\varepsilon,\varepsilon}$ be the solution of the discretised equation (5.3), driven by the noise $\xi^{\bar\varepsilon,\varepsilon}$, with the renormalisation constant $C^{(\bar\varepsilon,\varepsilon)} \stackrel{\text{def}}{=} 3C_1^{(\bar\varepsilon,\varepsilon)} - 9C_2^{(\bar\varepsilon,\varepsilon)}$, where $C_1^{(\bar\varepsilon,\varepsilon)}$ and $C_2^{(\bar\varepsilon,\varepsilon)}$ are given in (5.20). In what follows we will consider Φ^ε and $\Phi^{\bar\varepsilon,\varepsilon}$ as in Remark 4.3.7, built by the solutions to respective abstract problems with the discrete models $\hat Z^\varepsilon$ and $\hat Z^{\bar\varepsilon,\varepsilon}$ respectively, where the latter models are those from Lemma 5.4.1.

In order to have a priori bounds on the processes and models introduced above, we define for every $K > 0$ the following stopping times:

$$\sigma_K^\varepsilon \stackrel{\text{def}}{=} \inf\bigl\{T > 0 : \|\Phi\|_{\mathcal{C}^{\delta,\alpha}_{\bar\eta,T}} \geq K \;\text{ or }\; \|\hat Z\|_{\delta,\gamma;T} \geq K\;,\;\text{ or }\; \|\hat Z^\varepsilon\|^{(\varepsilon)}_{\delta,\gamma;T} \geq K\bigr\}\;,$$
$$\sigma^{\bar\varepsilon,\varepsilon} \stackrel{\text{def}}{=} \inf\bigl\{T > 0 : \|\Phi - \Phi^{\bar\varepsilon,0}\|_{\mathcal{C}^{\delta,\alpha}_{\bar\eta,T}} \geq 1 \;\text{ or }\; \|\Phi^\varepsilon - \Phi^{\bar\varepsilon,\varepsilon}\|^{(\varepsilon)}_{\mathcal{C}^{\delta,\alpha}_{\bar\eta,T}} \geq 1\;,$$
$$\text{or }\; \|\Phi^{\bar\varepsilon,0}; \Phi^{\bar\varepsilon,\varepsilon}\|^{(\varepsilon)}_{\mathcal{C}^{\delta,\alpha}_{\bar\eta,T}} \geq 1\;,\;\text{ or }\; \|\hat Z; \hat Z^{\bar\varepsilon,0}\|_{\delta,\gamma;T} \geq 1\;,\;\text{ or }\; \|\hat Z^\varepsilon; \hat Z^{\bar\varepsilon,\varepsilon}\|^{(\varepsilon)}_{\delta,\gamma;T} \geq 1\bigr\}\;,$$

as well as $\varrho_K^{\bar\varepsilon,\varepsilon} \stackrel{\text{def}}{=} \sigma_K^\varepsilon \wedge \sigma^{\bar\varepsilon,\varepsilon}$. Then, choosing two constants $\bar K > K$ and using the latter stopping time and the triangle inequality, we get the following bound:

$$\mathbb{P}\Bigl[\|\Phi; \Phi^\varepsilon\|^{(\varepsilon)}_{\mathcal{C}^{\delta,\alpha}_{\bar\eta,\tau_K}} \geq c\Bigr] \leq \mathbb{P}\Bigl[\|\Phi - \Phi^{\bar\varepsilon,0}\|_{\mathcal{C}^{\delta,\alpha}_{\bar\eta,\varrho^{\bar\varepsilon,\varepsilon}_{\bar K}}} \geq c\Bigr] + \mathbb{P}\Bigl[\|\Phi^{\bar\varepsilon,0}; \Phi^{\bar\varepsilon,\varepsilon}\|^{(\varepsilon)}_{\mathcal{C}^{\delta,\alpha}_{\bar\eta,\varrho^{\bar\varepsilon,\varepsilon}_{\bar K}}} \geq c\Bigr]$$
$$+ \mathbb{P}\Bigl[\|\Phi^{\bar\varepsilon,\varepsilon} - \Phi^\varepsilon\|^{(\varepsilon)}_{\mathcal{C}^{\delta,\alpha}_{\bar\eta,\varrho^{\bar\varepsilon,\varepsilon}_{\bar K}}} \geq c\Bigr] + \mathbb{P}\bigl[\varrho^{\bar\varepsilon,\varepsilon}_{\bar K} < \sigma^\varepsilon_{\bar K}\bigr] + \mathbb{P}\bigl[\sigma^\varepsilon_{\bar K} < \tau_K\bigr]\;. \quad (5.35)$$

We will show that if we take the limits $\varepsilon, \bar\varepsilon \to 0$ and $K, \bar K \to \infty$, then all the terms on the right-hand side of (5.35) vanish and we obtain the claim (5.34).

It follows from the definition of $\varrho^{\bar\varepsilon,\varepsilon}_{\bar K}$ that $\|\hat Z\|_{\delta,\gamma;\varrho^{\bar\varepsilon,\varepsilon}_{\bar K}}$ and $\|\hat Z^{\bar\varepsilon,0}\|_{\delta,\gamma;\varrho^{\bar\varepsilon,\varepsilon}_{\bar K}}$ are bounded by constants proportional to $\bar K$. Hence, Theorems 4.3.8 and 4.2.6, and Lemma 5.4.2 yield

$$\lim_{\bar\varepsilon\to 0} \mathbb{P}\Bigl[\|\Phi - \Phi^{\bar\varepsilon,0}\|_{\mathcal{C}^{\delta,\alpha}_{\bar\eta,\varrho^{\bar\varepsilon,\varepsilon}_{\bar K}}} \geq c\Bigr] = 0\;,$$

uniformly in ε. Similarly, we can use Theorems 4.3.8 and 4.2.6, and the bounds on discrete models from Lemma 5.4.1 to obtain the uniform in ε convergence

$$\lim_{\bar\varepsilon\to 0} \mathbb{P}\Bigl[\|\Phi^\varepsilon - \Phi^{\bar\varepsilon,\varepsilon}\|^{(\varepsilon)}_{\mathcal{C}^{\delta,\alpha}_{\bar\eta,\varrho^{\bar\varepsilon,\varepsilon}_{\bar K}}} \geq c\Bigr] = 0\;.$$

Now, we turn to the second term in (5.35). It follows from our definitions that we have $\xi^{\bar\varepsilon,\varepsilon} = \varrho^{\bar\varepsilon,\varepsilon} \star \xi$, where

$$\varrho^{\bar\varepsilon,\varepsilon}(t,x) \stackrel{\text{def}}{=} \varepsilon^{-d} \int_{\Lambda_\varepsilon^d} \psi^{\bar\varepsilon,\varepsilon}(t,y)\, \mathbf{1}_{|y-x|_\infty \leq \varepsilon/2}\, dy\;.$$

Moreover, for $z = (t,x) \in \mathbb{R}\times\Lambda_\varepsilon^d$ one has the identity

$$\bigl(\psi^{\bar\varepsilon} - \varrho^{\bar\varepsilon,\varepsilon}\bigr)(z) = \varepsilon^{-2d} \int_{\Lambda_\varepsilon^d}\int_{\mathbb{R}^d} (\psi^{\bar\varepsilon}(t,x) - \psi^{\bar\varepsilon}(t,u))\mathbf{1}_{|u-y|_\infty\leq\varepsilon/2}\mathbf{1}_{|y-x|_\infty\leq\varepsilon/2}\, du\, dy\;,$$

from which we immediately obtain the bound

$$\sup_{z\in\mathbb{R}\times\Lambda_\varepsilon^d}\big|D_t^k\big(\psi^{\bar\varepsilon}-\varrho^{\bar\varepsilon,\varepsilon}\big)(z)\big| \lesssim \varepsilon\bar\varepsilon^{-|\mathfrak{s}|-k\mathfrak{s}_0-1}\;,$$

for every $k\in\mathbb{N}$. Hence, using the a priori bounds on the solutions, which follow from the definition of $\varrho_{\bar K}^{\bar\varepsilon,\varepsilon}$, we can use the standard result from numerical analysis of ODEs (see e.g. [Lam91]) that the second term in (5.35) vanishes as $\varepsilon\to 0$, as soon as $\bar\varepsilon$ is fixed.

The limit $\lim_{\bar\varepsilon\to0}\lim_{\varepsilon\to0}\mathbb{P}\big[\varrho_{\bar K}^{\bar\varepsilon,\varepsilon} < \sigma_{\bar K}^\varepsilon\big] = 0$ follows immediately from the definition of the involved stopping times, Lemmas 5.4.1, 5.4.2 and the convergences we have just proved. Finally, it follows from Lemma 5.4.1 that

$$\lim_{\bar K\to\infty}\mathbb{P}\big[\sigma_{\bar K}^\varepsilon < \tau_K\big] = 0\;,$$

for a fixed K and uniformly in ε, which finishes the proof. $\square$

Proof of Corollary 5.1.2. Let ξ be space-time white noise on some probability space $(\Omega,\mathscr{F},\mathbf{P})$, and let its discretisation ξ^ε be given by (5.4). Let furthermore Φ_0^ε be a random variable on the same probability space which is independent of ξ and such that the solution to (5.3) with the nearest neighbours approximate Laplacian Δ^ε and driven by ξ^ε is stationary. We denote by μ_ε its stationary distribution, which we view as a measure on $\mathcal{C}^\alpha$ with α as in (5.5), by extending it in a piecewise constant fashion. It then follows from [BFS83] (by combining Eq. 8.2 and Thm. 6.1 in that article) that the sequence μ_ε is tight in $\mathcal{C}^\alpha$ as $\varepsilon\to0$ with uniformly bounded moments of all orders, so we can choose a subsequence (which we also denote by μ_ε), weakly converging to an accumulation point μ. Actually, combining this with [Par77] (see also [Par75]) shows that μ is unique and coincides with the Φ_3^4 measure constructed in [Fel74]. In particular, if we view Φ_0^ε as an element of $\mathcal{C}^\alpha$ by piecewise constant extension, we can and will assume by Skorokhod's representation theorem that Φ_0^ε converges almost surely as $\varepsilon\to0$ to a limit $\Phi_0\in\mathcal{C}^\alpha$. In order to use Skorokhod's representation theorem [Kal02], the underlying spaces have to be separable which doesn't hold for $\mathcal{C}^\alpha$. Nevertheless, we can use the fact that the random variables belong almost surely to the closure of smooth functions under the seminorm (3.3) which is separable.

Before we proceed, we introduce the space $\bar{\mathcal{C}} \stackrel{\text{def}}{=} \mathcal{C}_{\bar\eta}^{0,\alpha}\big([0,1],\mathbb{T}^3\big)\cup\{\infty\}$ (the latter Hölder space is a subspace of $\mathcal{C}^{0,\alpha}_{\bar\eta}\big([0,1],\mathbb{R}^3\big)$ defined below (3.5), containing the spatially periodic distributions), for α and $\bar\eta$ as in (5.5), and equipped with the metric such that

$$\begin{aligned}
&d(\zeta,\infty) \stackrel{\text{def}}{=} d(\infty,\zeta) \stackrel{\text{def}}{=} \big(1+\|\zeta\|_{\mathcal{C}^{0,\alpha}_{\bar\eta,1}}\big)^{-1}\;, \quad \zeta\neq\infty\;,\\
&d(\zeta_1,\zeta_2) \stackrel{\text{def}}{=} \min\big\{\|\zeta_1-\zeta_2\|_{\mathcal{C}^{0,\alpha}_{\bar\eta,1}}, d(\zeta_1,\infty)+d(\zeta_2,\infty)\big\}\;, \quad \zeta_i\neq\infty\;.
\end{aligned}$$

Denote now by Φ^ε the solution to (5.3) with initial condition Φ^ε_0 and by Φ the solution to (3.2) with initial condition Φ_0. We can view these as $\bar{\mathcal{C}}$-valued random variables by postulating that $\Phi=\infty$ if its lifetime is smaller than 1. (The lifetime of Φ^ε is always infinite for fixed ε.)

Since the assumptions of Theorem 5.1.1 are fulfilled, the convergence (5.5) holds and, since solutions blow up at time T^*, this implies that $d(\Phi^\varepsilon,\Phi)\to 0$ in probability, as $\varepsilon\to 0$. (The required continuity in time obviously holds for every Φ^ε and Φ.) In order to conclude, it remains to show that $\mathbb{P}(\Phi=\infty)=0$. In particular, since the only point of discontinuity of the evaluation maps $\Phi\mapsto\Phi(t,\cdot)$ on $\bar{\mathcal{C}}$ is ∞, this would then immediately show not only that solutions Φ live up to time 1 (and therefore any time) almost surely, but also that μ is invariant for Φ. To show that $\Phi\neq\infty$ a.s., it suffices to prove that there is no atom of the measure μ at the point ∞. Precisely, our aim is to show that for every $\bar\varepsilon>0$ there exists a constant $C_{\bar\varepsilon}>0$ such that

$$\mathbb{P}\Big(\|\Phi^\varepsilon\|_{\mathcal{C}^{0,\alpha}_{\bar\eta,1}}\geq C_{\bar\varepsilon}\Big)\leq\bar\varepsilon\;. \tag{5.36}$$

We fix $\bar\varepsilon>0$ in what follows and work with a generic constant $C_{\bar\varepsilon}>0$, whose value will be chosen later. For integers $K\geq 2$ and $i\in\{0,\dots,K-2\}$, we denote

$$Q^\varepsilon_{K,i} \stackrel{\text{def}}{=} \|\Phi^\varepsilon\|_{\mathcal{C}^{0,\alpha}_{\bar\eta,[i/K,(i+2)/K]}}\;,$$

where the norm $\|\cdot\|_{\mathcal{C}^{0,\alpha}_{\bar\eta,[T_1,T_2]}}$ is defined as below (3.5), but on the time interval $[T_1,T_2]$ and with a blow-up at T_1. Splitting the time interval $(0,1]$ in (3.5) into subintervals of length $1/K$, and deriving estimates on each subinterval, one gets

$$\|\Phi^\varepsilon\|_{\mathcal{C}^{0,\alpha}_{\bar\eta,1}}\leq Q^\varepsilon_{K,0}+\sum_{i=1}^{K-1}(i+1)^{-\bar\eta/2}\,Q^\varepsilon_{K,i-1}\leq\tilde C K^{-\bar\eta/2}\sum_{i=0}^{K-2}Q^\varepsilon_{K,i}\;,$$

if $\bar\eta \le 0$, and for some $\tilde C$ independent of K and ε. Since, by stationarity, the random variables $Q^\varepsilon_{K,i}$ all have the same law, it follows that

$$\begin{aligned}\mathbb{P}\Big(\|\Phi^\varepsilon\|_{\mathcal{C}^{0,\alpha}_{\bar\eta,1}} \ge C_{\bar\varepsilon}\Big) &\le \mathbb{P}\Big(\tilde C K^{-\bar\eta/2}\sum_{i=0}^{K-2} Q^\varepsilon_{K,i} \ge C_{\bar\varepsilon}\Big)\\ &\le K\mathbb{P}\Big(\|\Phi^\varepsilon\|_{\mathcal{C}^{0,\alpha}_{\bar\eta,2/K}} \ge \tilde C^{-1}K^{\bar\eta/2}C_{\bar\varepsilon}\Big)\;, \end{aligned}\tag{5.37}$$

To make the notation concise, we write $\tilde C_{K,\bar\varepsilon} \stackrel{\text{def}}{=} \tilde C^{-1}K^{\bar\eta/2}C_{\bar\varepsilon}$. Furthermore, in order to have a uniform bound on the initial data and the model, we use the following estimate

$$\begin{aligned}\mathbf{P}\Big(\|\Phi^\varepsilon\|_{\mathcal{C}^{0,\alpha}_{\bar\eta,2/K}} \ge \tilde C_{K,\bar\varepsilon}\Big) &\le \mathbf{P}\Big(\|\Phi^\varepsilon\|_{\mathcal{C}^{0,\alpha}_{\bar\eta,2/K}} \ge \tilde C_{K,\bar\varepsilon}\Big|\|\Phi^\varepsilon_0\|_{\mathcal{C}^\eta} \le L, \|\!|\!|Z^\varepsilon|\!|\!|^{(\varepsilon)}_{\gamma;1} \le L\Big)\\ &\quad+\mathbf{P}\Big(\|\Phi^\varepsilon_0\|_{\mathcal{C}^\eta} > L\Big) + \mathbf{P}\Big(\|\!|\!|Z^\varepsilon|\!|\!|^{(\varepsilon)}_{\gamma;1} > L\Big),\end{aligned}\tag{5.38}$$

valid for every L, where η and $\gamma > 0$ are as in the proof of Theorem 5.1.1.

Recalling that [BFS83, Sec. 8] yields uniform bounds on all moments of μ_ε, and using the first bound in (5.21), Markov's inequality implies that

$$\mathbf{P}\Big(\|\Phi^\varepsilon_0\|_{\mathcal{C}^\eta} > L\Big) \le B_1 L^{-q}\;,\qquad \mathbf{P}\Big(\|\!|\!|Z^\varepsilon|\!|\!|^{(\varepsilon)}_{\gamma;1} > L\Big) \le B_2 L^{-q}\;,\tag{5.39}$$

for any $q \ge 1$, and for constant B_1 and B_2 independent of ε and L.

Turning to the first term in (5.38), it follows from the fixed point argument in the proof of Theorem 4.3.8 and the bound (4.8a), that there exists $\tilde p \ge 1$ such that one has the bound

$$\|\Phi^\varepsilon\|_{\mathcal{C}^{0,\alpha}_{\bar\eta,2/K}} \le B_3 L^3\;,$$

with B_3 being independent of ε and L, as soon as $\|\Phi^\varepsilon_0\|_{\mathcal{C}^\eta} \le L$, $\|\!|\!|Z^\varepsilon|\!|\!|^{(\varepsilon)}_{\gamma;1} \le L$, $K \ge L^{\tilde p}$ and $L \ge 2$. In particular, the first term vanishes if we can ensure that

$$\tilde C_{K,\bar\varepsilon} \ge B_3 L^3\;.\tag{5.40}$$

Choosing first L large enough so that the contribution of the two terms in (5.39) is smaller than $\bar\varepsilon/2$, then K large enough so that $K \ge L^{\tilde p}$, and finally $C_{\bar\varepsilon}$ large enough so that (5.40) holds, the claim follows.

The fact that Φ is a Markov process and the measure μ is reversible for it, follows immediately from the same properties of the discretised equation and convergence (5.5). □

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